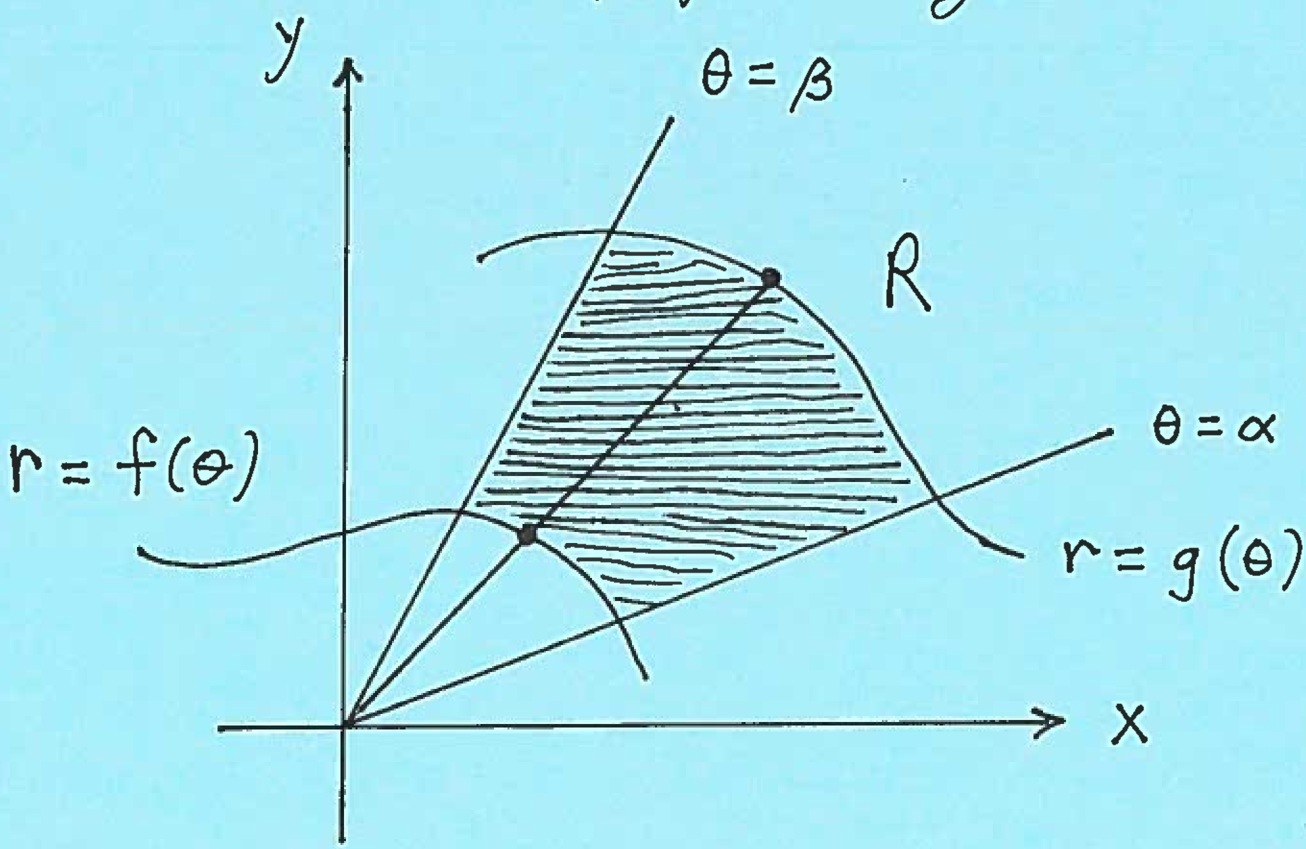
Section 15.3
Thomas Calculus
11th Ed.

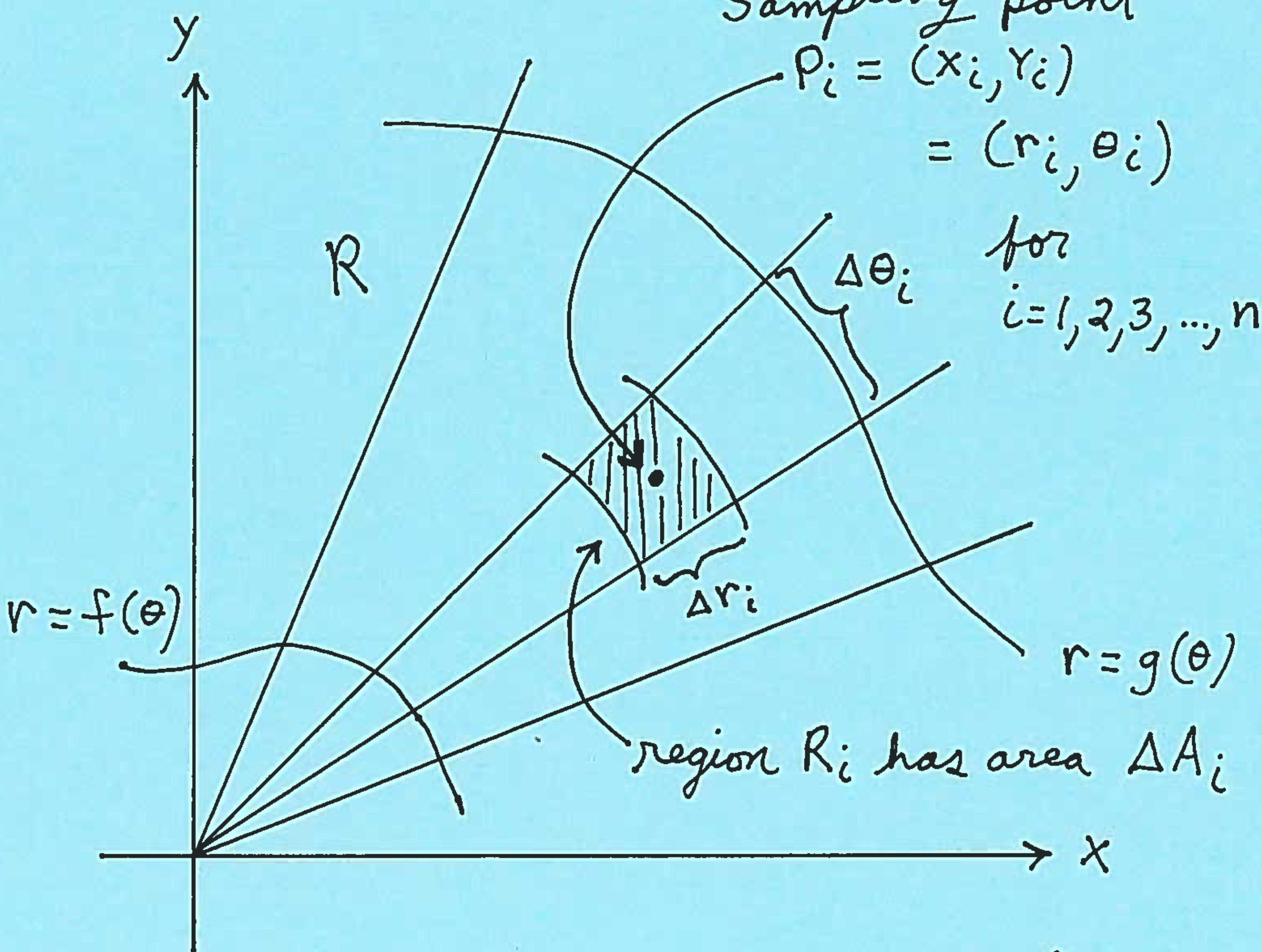
Double Integrals Using Polar Coordinates

Consider surface $z = f(x, y)$ over region R given by

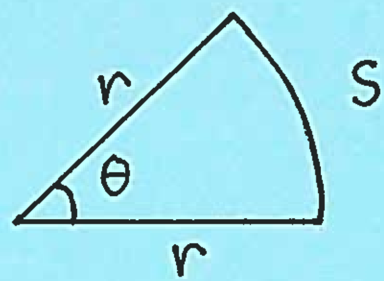
$$R: \begin{cases} \alpha \le \theta \le \beta \\ f(\theta) \le r \le g(\theta) \end{cases}$$



Partition region R in the following manner:



<u>Recall</u>: (circular arc length)



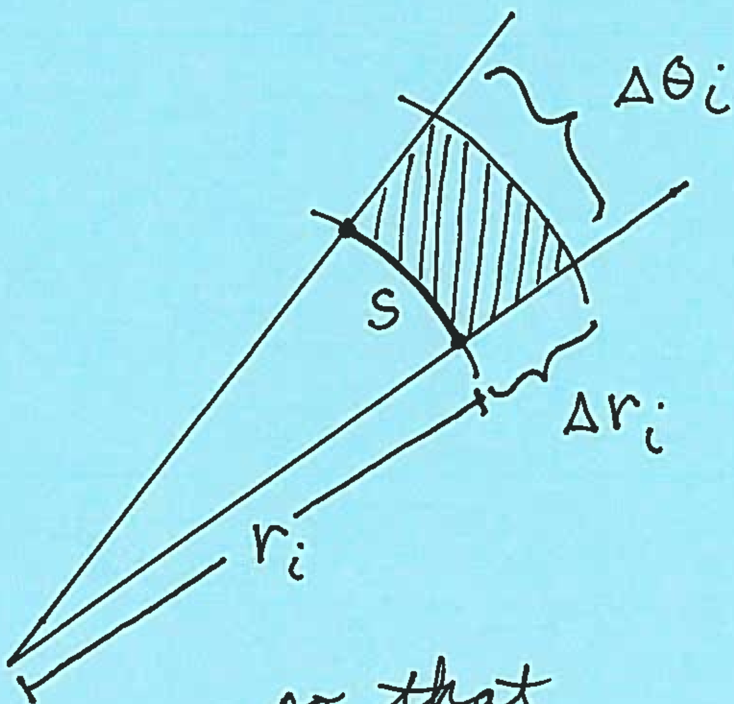
circular arc length S subtended by angle θ radians is:

$S = \left(\begin{matrix}\text{fraction of} \\ \text{circumference}\end{matrix}\right) \cdot (\text{circumference})$

$= \left(\frac{\theta}{\cancel{2\pi}}\right)(\cancel{2\pi} r) = r\theta$, i.e.,

$\boxed{S = r\theta}$; then

region R_i ($\approx$ rectangle) has area



$$\Delta A_i = (s)(\Delta r_i)$$

$$= r_i(\Delta\theta_i)(\Delta r_i)$$

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so that

$$\iint_R f(P)\, dA = \lim_{\text{mesh}\to 0} \sum_{i=1}^{n} f(P_i)\cdot \Delta A_i$$

$$= \lim_{\text{mesh}\to 0} \sum_{i=1}^{n} f(r_i, \theta_i)\cdot r_i(\Delta r_i)(\Delta\theta_i)$$

$$= \int_{\alpha}^{\beta} \int_{f(\theta)}^{g(\theta)} f(r,\theta)\cdot r \, dr \, d\theta \,, \quad i.e.,$$

$$\boxed{\iint_R f(P)\, dA = \int_{\alpha}^{\beta} \int_{f(\theta)}^{g(\theta)} f(r,\theta)\cdot r \, dr \, d\theta}$$

Example: Evaluate the integral.

$$\int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} 4 \sin\theta \cdot r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(4 \sin\theta \cdot \frac{1}{2} r^2\right)\Big|_{r=0}^{r=\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2 \sin\theta \, (\cos\theta)^2 \, d\theta$$

$$= -\frac{2}{3}(\cos\theta)^3 \Big|_0^{\frac{\pi}{2}} = -\frac{2}{3}(\overset{0}{\cos\tfrac{\pi}{2}})^3 - -\frac{2}{3}(\overset{1}{\cos 0})^3$$

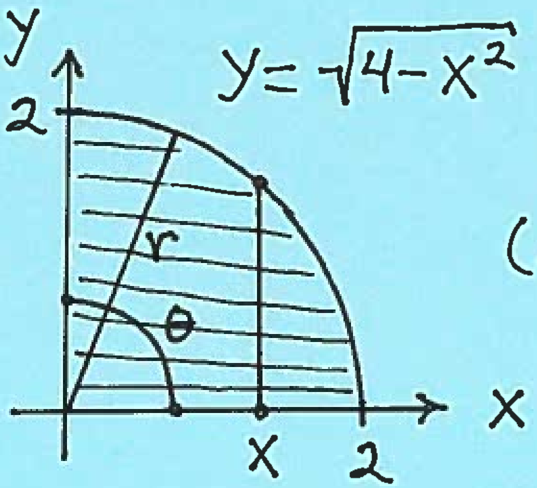
$$= \frac{2}{3}$$

Question: When would we want to use polar coordinates?

Answer: See the next example.

Example: Convert the very difficult Double Integral in rectangular coordinates to polar coordinates, and evaluate it.

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$



$$R: \begin{cases} 0 \le x \le 2 \\ 0 \le y \le \sqrt{4-x^2} \end{cases}$$

(rectangular coordinates)

$$R: \begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2 \end{cases}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \sqrt{r^2} \cdot r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 r^2 \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3} r^3 \Big|_{r=0}^{r=2} \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{8}{3} \, d\theta$$

$$= \frac{8}{3}\theta \Big|_0^{\frac{\pi}{2}} = \frac{8}{3}\left(\frac{\pi}{2}\right) = \frac{4}{3}\pi .$$

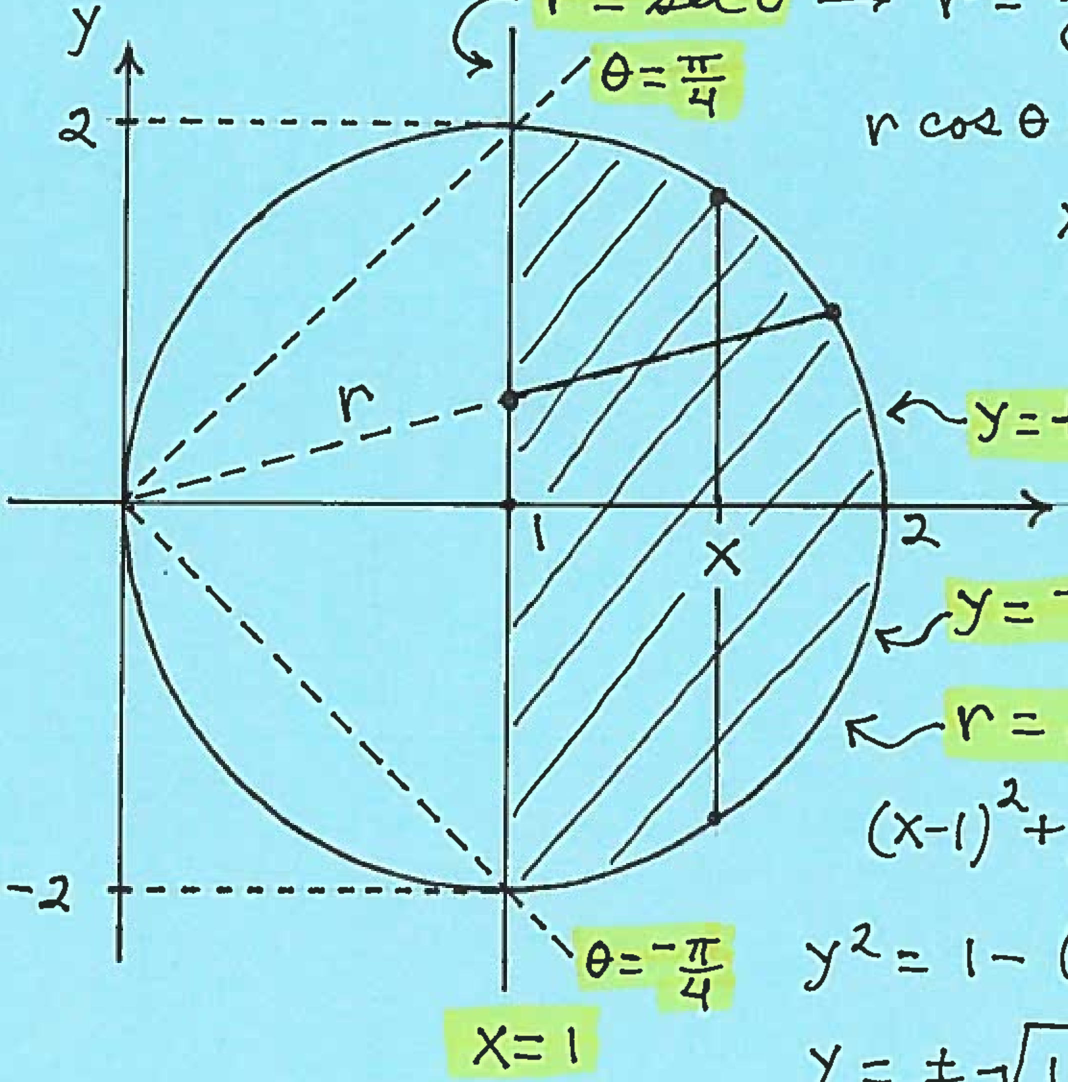
$\boxed{\underline{\text{Example}}}$: Consider region R inside the circle $r = 2\cos\theta$ and to the right of the line $r = \sec\theta$. Find the Area of R using a.) polar coordinates.

b.) rectangular coordinates.

$r = \sec\theta \rightarrow r = \dfrac{1}{\cos\theta} \rightarrow$

$r\cos\theta = 1 \rightarrow$

$x = 1$



$y = +\sqrt{1-(x-1)^2}$

$y = -\sqrt{1-(x-1)^2}$

$r = 2\cos\theta \rightarrow$

$(x-1)^2 + y^2 = 1^2 \rightarrow$

$y^2 = 1-(x-1)^2 \rightarrow$

$y = \pm\sqrt{1-(x-1)^2}$

(SET UP ONLY)

$R: \begin{cases} -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \\ \sec\theta \leq r \leq 2\cos\theta \end{cases}$; $R: \begin{cases} 1 \leq x \leq 2 \\ \uparrow \\ -\sqrt{1-(x-1)^2} \leq y \leq +\sqrt{1-(x-1)^2} \end{cases}$

a.) $\text{Area} = \iint_R 1 \, dA$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} r \, dr \, d\theta$$

b.) $\text{Area} = \iint_R 1 \, dA$

$$= \int_1^2 \int_{-\sqrt{1-(x-1)^2}}^{+\sqrt{1-(x-1)^2}} 1 \, dy \, dx$$

Example: Convert the following Double Integral in polar coordinates to a Double Integral in rectangular coordinates.

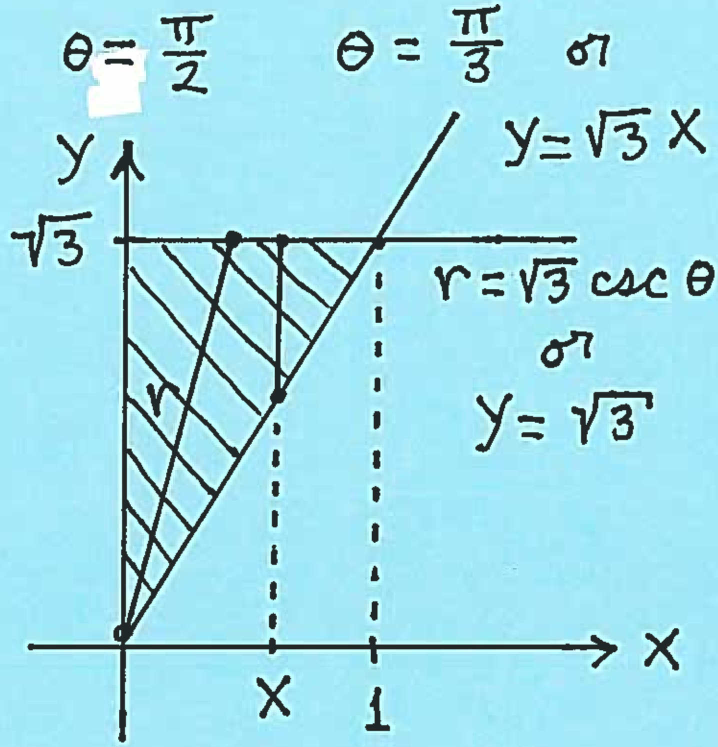
$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{\sqrt{3}\csc\theta} r^4 \sin^2\theta \cos\theta \, dr \, d\theta$$

$$r = \sqrt{3}\,\csc\theta \rightarrow$$

$$r = \sqrt{3} \cdot \frac{1}{\sin\theta} \rightarrow$$

$$r\sin\theta = \sqrt{3} \rightarrow$$

$$y = \sqrt{3}$$



$$R: \begin{cases} \frac{\pi}{3} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le \sqrt{3}\,\csc\theta \end{cases} \quad \text{and}$$

$$R: \begin{cases} 0 \le x \le 1 \\ \sqrt{3}\,x \le y \le \sqrt{3} \end{cases}$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{0}^{\sqrt{3}\csc\theta} (r\cos\theta)(r\sin\theta)^2 \cdot r \, dr \, d\theta$$

$$= \int_{0}^{1} \int_{\sqrt{3}x}^{\sqrt{3}} x\,y^2 \, dy \, dx$$