Section 16.2
Thomas Calculus
11th Ed.

Vector Fields and Flow along a Path C

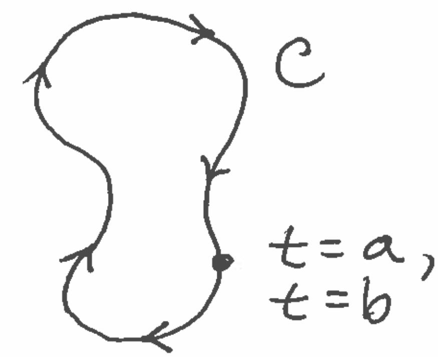
Definition: Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ be vector field of velocity vectors (representing mass, momentum, heat, etc.). Let $C: \vec{r}(t)$ for $a \le t \le b$ be curve in space on which $\vec{F}$ is defined. Let $\vec{T}$ be the unit tangent vector to C. Then the $\text{proj}_{\vec{T}} \vec{F}$ has scalar component $\vec{F} \cdot \vec{T}$, then the flow of $\vec{F}$ along path C is

$$\boxed{\text{Flow} = \int_C \vec{F} \cdot \vec{T} \, ds} .$$

Note: 1.) The Line Integral for

Flow is the same as for Work, but the UNITS on the final answers are different.

2.) If path C is a closed curve, then the Flow is called Circulation and is written $\oint_C \vec{F} \cdot \vec{T}\, ds$.



3.) If we assume that $\vec{F} = \delta \vec{V}$, where δ is the density of the fluid (mass/area units) and $\vec{V}$ is the velocity vector for the fluid, then the Flow units will be

$$\underbrace{\left(\frac{\text{mass}}{\text{area}} \cdot \frac{\text{length}}{\text{time}}\right)}_{\vec{F}} \times \underbrace{\left(\frac{\text{length}}{\text{time}}\right)}_{\cdot\ \vec{r}\,'(t)} \times \underbrace{(\text{time})}_{dt}$$

$= \dfrac{\text{mass}}{\text{time}}$, and is called the <u>mass</u> <u>flow</u> <u>rate</u>.

<u>Example</u>: Consider the velocity field

$$\vec{F}(x,y) = (y^2)\vec{i} + (x^2)\vec{j}$$

of a fluid with density units $gm/cm.^2$ Find the Flow of $\vec{F}$ along path C given by

$$C: \vec{r}(t) = (2t)\vec{i} + (t^2)\vec{j}$$

for $0 \le t \le 2$ seconds.

$\vec{r}(t) = (2t)\vec{i} + (t^2)\vec{j} \xrightarrow{D} \vec{v}(t) = (2)\vec{i} + (2t)\vec{j}$,

and $\vec{F}$ along C becomes

$$\vec{F} = (t^2)^2\vec{i} + (2t)^2\vec{j} = (t^4)\vec{i} + (4t^2)\vec{j};$$

the Flow of $\vec{F}$ along path C is

$$\text{Flow} = \int_C \vec{F} \cdot \vec{v}(t)\, dt$$

$$= \int_0^2 \left[(2)(t^4) + (2t)(4t^2)\right] dt$$

$$= \int_0^2 \left[2t^4 + 8t^3\right] dt$$

$$= \left(\frac{2}{5}t^5 + 2t^4\right)\Big|_0^2$$

$$= \frac{64}{5} + 32 = \frac{64}{5} + \frac{160}{5} = \frac{224}{5} \ \frac{\text{gm.}}{\text{sec.}}$$

<u>Example</u>: Consider the velocity field

$$\vec{F}(x,y,z) = x\vec{i} + z\vec{j} + y\vec{k}$$

of a fluid with density units kg./m^2. Find the Flow of $\vec{F}$ along path C given by

$$C: \vec{r}(t) = (e^{2t})\vec{i} + (e^{t})\vec{j} + (e^{-t})\vec{k}$$

for $0 \le t \le \ln 2$ minutes.

$\vec{r}(t) = (e^{2t})\vec{i} + (e^{t})\vec{j} + (e^{-t})\vec{k} \quad \xrightarrow{D}$

$\vec{v}(t) = (2e^{2t})\vec{i} + (e^{t})\vec{j} + (-e^{-t})\vec{k}$, and

$\vec{F}$ along C becomes

$\vec{F} = (e^{2t})\vec{i} + (e^{-t})\vec{j} + (e^{t})\vec{k}$;

the Flow of $\vec{F}$ along path C is

$$\text{Flow} = \int_C \vec{F} \cdot \vec{v}(t)\, dt$$

$$= \int_0^{\ln 2} \left[(e^{2t})(2e^{2t}) + (e^{-t})(e^{t}) + (e^{t})(-e^{-t})\right] dt$$

$$= \int_0^{\ln 2} \left[2e^{4t} + 1 - 1\right] dt$$

$$= 2 \cdot \frac{1}{4} e^{4t} \Big|_0^{\ln 2}$$

$$= \frac{1}{2} e^{4 \ln 2} - \frac{1}{2} e^{0}$$

$$= \frac{1}{2} e^{\ln 2^4} - \frac{1}{2}(1)$$

$$= \frac{1}{2}(16) - \frac{1}{2} = \frac{15}{2} \quad \frac{\text{kg.}}{\text{min.}}$$

<u>Example</u>: (Alternate Notation for Work/Flow Line Integrals) Evaluate the Line Integral

$$\int_C \sqrt{z}\, dx + \sqrt{x}\, dy + y^2\, dz,$$

where path C is given by

C: $x = t$, $y = t^{3/2}$, $z = t^2$

for $0 \le t \le 2$.

$$\int_C \sqrt{z}\, dx + \sqrt{x}\, dy + y^2 dz$$

$$= \int_0^2 \left(\sqrt{z} \cdot \frac{dx}{dt} + \sqrt{x} \cdot \frac{dy}{dt} + y^2 \cdot \frac{dz}{dt} \right) dt$$

$$= \int_0^2 \left(\sqrt{t^2} \cdot (1) + \sqrt{t} \left(\frac{3}{2} t^{1/2} \right) + \left(t^{3/2} \right)^2 (2t) \right) dt$$

$$= \int_0^2 \left(t + \frac{3}{2} t + 2t^4 \right) dt$$

$$= \int_0^2 \left(\frac{5}{2} t + 2t^4 \right) dt$$

$$= \left(\frac{5}{2} \cdot \frac{1}{2} t^2 + \frac{2}{5} t^5 \right) \Big|_0^2$$

$$= 5 + \frac{64}{5}$$

$$= \frac{25}{5} + \frac{64}{5}$$

$$= \frac{89}{5}$$