Math 21D
Kouba
Vector Fields, Work, Flow, Flux

Vector Fields

Ex: Assume that water flows through a space and at each point $P = (x, y, z)$ we know the speed and direction of water flow:

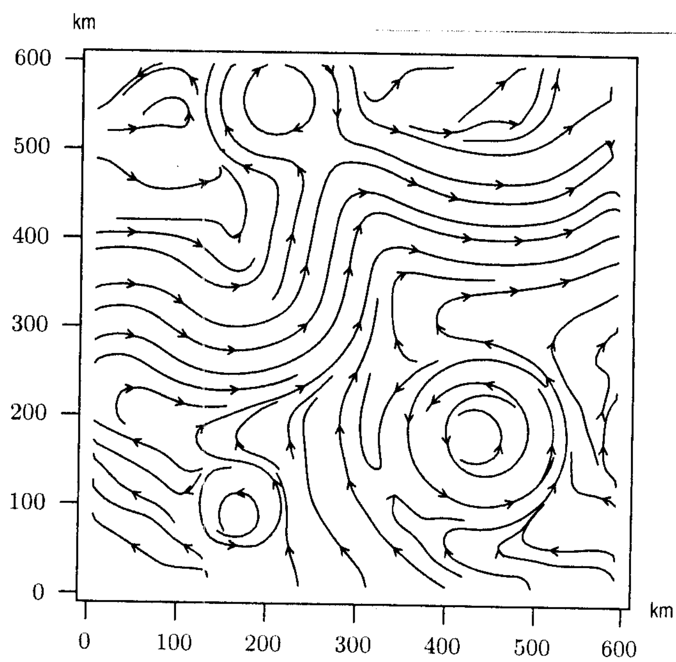


Figure 17.32: Flow lines for objects in the Gulf stream with different starting points
(Calculus by Hughes-Hallett, 3rd. edition, p. 805)

We call this representation a vector field.

Def: A vector field is a function which

assigns a vector to each point in space (also called force field, velocity field, gradient field, gravitational field, etc.)

1.) (2D-space)

$$\vec{F}(x,y) = M(x,y)\cdot\vec{i} + N(x,y)\,\vec{j}$$

2.) (3D-space)

$$\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$$

<u>Ex</u>: The following are vector fields.

1.) $\vec{F}(x,y) = (xy)\,\vec{i} + (x^2+y^2)\,\vec{j}$

2.) $\vec{F}(x,y,z) = (\cos x)\,\vec{i} + (\sin y)\,\vec{j} + \tan(xy)\cdot\vec{k}$

<u>Def</u>: The <u>gradient field</u> of scalar function $f(x,y,z)$ is the vector field

$$\vec{\nabla} f = f_x\cdot\vec{i} + f_y\cdot\vec{j} + f_z\cdot\vec{k}.$$

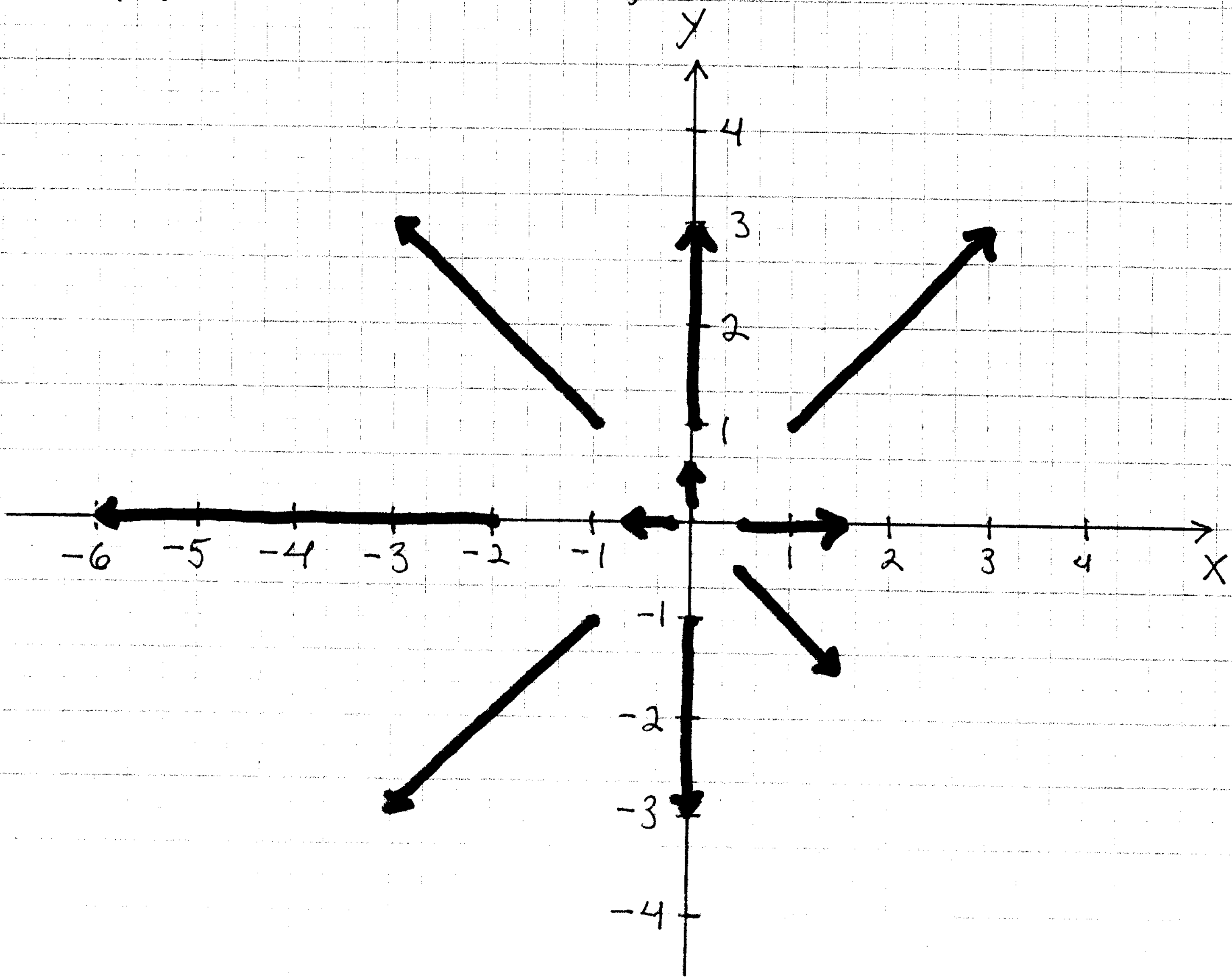
<u>Ex</u>: Determine the gradient field for $f(x,y) = x^2+y^2$ and plot gradient vectors for the following points:

$(\frac{1}{2},0)$, $(1,1)$, $(0,1)$, $(-1,1)$, $(-\frac{1}{4},0)$, $(-2,0)$, $(-1,-1)$, $(0,-1)$, $(\frac{1}{2},\frac{-1}{2})$, and $(0,\frac{1}{4})$

$$\vec{\nabla} f = f_x \cdot \vec{i} + f_y \cdot \vec{j} \longrightarrow$$

$$\vec{\nabla} f(x,y) = (2x)\,\vec{i} + (2y)\,\vec{j} \quad ;$$

(x,y)	$\vec{\nabla} f(x,y)$	(x,y)	$\vec{\nabla} f(x,y)$
$(\frac{1}{2},0)$	$\vec{i}$	$(-2,0)$	$-4\vec{i}$
$(1,1)$	$2\vec{i}+2\vec{j}$	$(-1,-1)$	$-2\vec{i}-2\vec{j}$
$(0,1)$	$2\vec{j}$	$(0,-1)$	$-2\vec{j}$
$(-1,1)$	$-2\vec{i}+2\vec{j}$	$(\frac{1}{2},\frac{-1}{2})$	$\vec{i}-\vec{j}$
$(-\frac{1}{4},0)$	$-\frac{1}{2}\vec{i}$	$(0,1/4)$	$\frac{1}{2}\vec{j}$

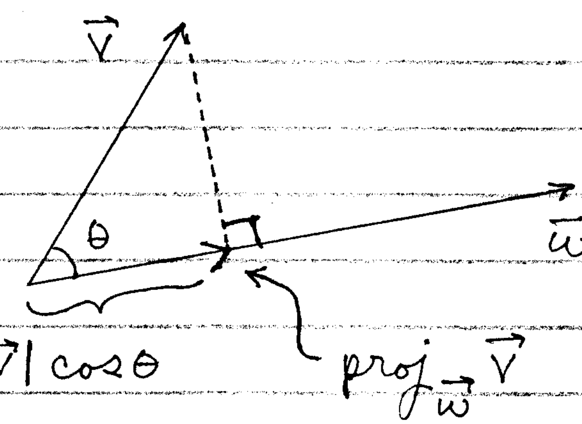


<u>Recall</u>: Let $z = f(x,y)$ be a surface in 3D-space. Then the gradient vector

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}$$

1.) is $\perp$ to the level curve $f(x,y) = c$.
2.) points in the direction of the maximum directional derivative.
3.) has magnitude equal to the value of the maximum directional derivative.

<u>Recall</u>: Let $\vec{v}$ and $\vec{w}$ be vectors. The <u>projection</u> of $\vec{v}$ onto $\vec{w}$ is



$$\text{proj}_{\vec{w}}\vec{v} = |\vec{v}|\cos\theta \cdot \frac{\vec{w}}{|\vec{w}|}$$

$$= \cancel{|\vec{v}|}\left(\frac{\vec{v}\cdot\vec{w}}{\cancel{|\vec{v}|}\,|\vec{w}|}\right)\frac{\vec{w}}{|\vec{w}|}$$

$$= \left(\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\right)\vec{w} \text{, i.e.,}$$

$$\text{proj}_{\vec{w}}\vec{v} = \left(\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\right)\vec{w}$$

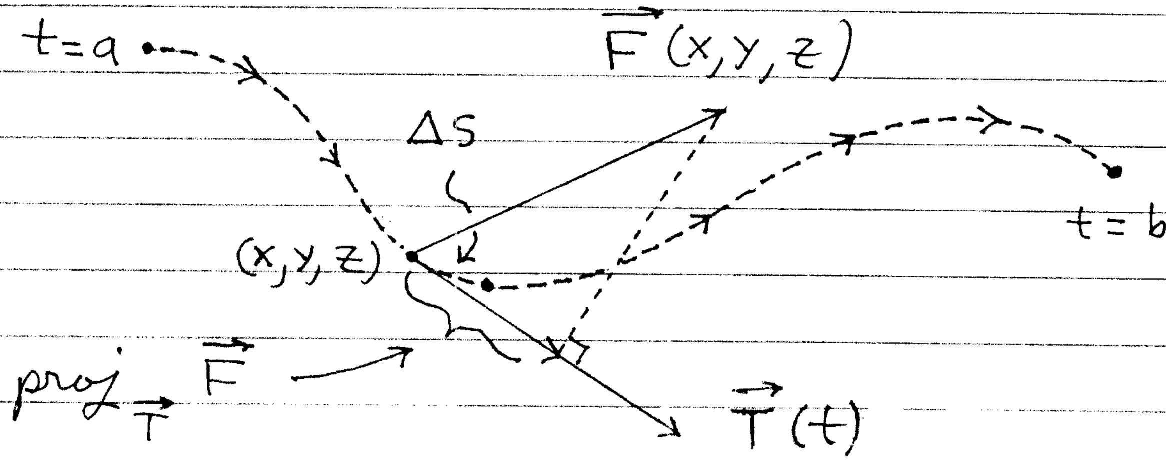
; if $\vec{w}$ is a unit vector, then $|\vec{w}| = 1$ and $\boxed{\text{proj}_{\vec{w}}\vec{v} = (\vec{v}\cdot\vec{w})\,\vec{w}}$.

We call $(\vec{v} \cdot \vec{w})$ the scalar component of $\vec{v}$ in the direction of $\vec{w}$.

Def: Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ be a vector field of force vectors (force field). Let $C: \vec{r}(t)$ for $a \le t \le b$ be a curve in space on which $\vec{F}$ is defined. We want a formula for the work required by this force field to push a particle along path C.

(Recall: Work = Force × Distance)

Let $\vec{T}(t)$ be the unit tangent vector to C. Then



has scalar component $\vec{F} \cdot \vec{T}$, so that the work done by $\vec{F}$ on $\vec{r}(t)$ is

$$\boxed{\text{Work} = \int_C \underbrace{\vec{F} \cdot \vec{T}}_{\text{force}} \; \underbrace{ds}_{\text{distance}}}.$$

Method of Computation :

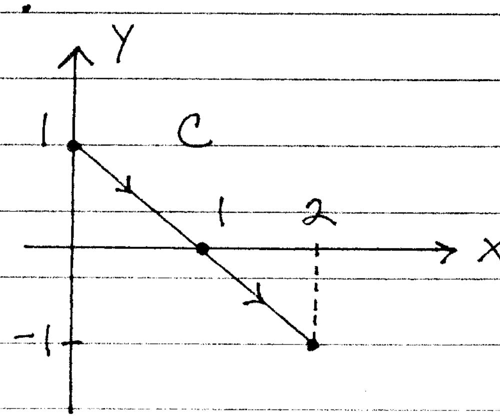
$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot \vec{T}\, \frac{ds}{dt}\, dt$$

$$= \int_C \vec{F} \cdot \frac{\vec{v}(t)}{\cancel{|\vec{v}(t)|}}\, \cancel{\frac{ds}{dt}}\, dt \qquad \left(|\vec{v}(t)| = \frac{ds}{dt} \right)$$

$$= \int_C \vec{F} \cdot \vec{v}(t)\, dt$$

$$= \int_a^b \vec{F} \cdot \vec{r}\,'(t)\, dt \qquad \left(\vec{v}(t) = \vec{r}\,'(t) \right)$$

Ex : Compute the work done by the force field $\vec{F}(x,y) = (2x)\,\vec{i} + (2y)\,\vec{j}$ along path C determined by $y = 1 - x$ for $0 \le x \le 2$.



$$C: \begin{cases} x = t \\ y = 1 - t \end{cases} \text{ for } 0 \le t \le 2$$

$$\vec{r}(t) = (t)\,\vec{i} + (1-t)\,\vec{j} \xrightarrow{D}$$

$$\vec{r}\,'(t) = (1)\,\vec{i} + (-1)\,\vec{j} \;,$$

$$\vec{F}(x,y) = \vec{F}(x(t), y(t)) = (2t)\,\vec{i} + (2-2t)\,\vec{j} \;;$$

then

$$\text{Work} = \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_C \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^2 \left[(2t)(1) + (2-2t)(-1)\right] dt$$

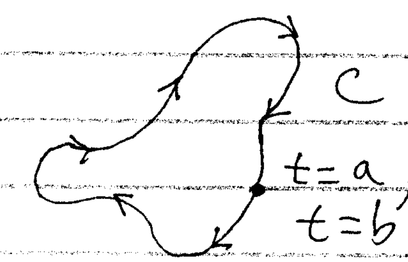
$$= \int_0^2 (4t-2)\, dt$$

$$= (2t^2 - 2t)\Big|_0^2 = 8 - 4 = 4$$

<u>Def</u>: If $\vec{F}$ is a velocity field (for fluid flows), the <u>flow</u> along the curve $C: \vec{r}(t)$ for $a \le t \le b$ is

$$\text{Flow} = \int_C \vec{F} \cdot \vec{T}\, ds$$

(If C is a closed curve this flow is also called <u>circulation</u> around the curve.)



<u>Note</u>: If we assume that $\vec{F} = \delta \vec{v}$, where δ is density of fluid (mass/area units) and $\vec{v}$ is velocity vector for fluid, then Flow is measured in $\frac{\text{mass}}{\text{time}}$ units.

Notation for Work Integral

Assume that vector field
$\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$
and path C is given by
$\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,k$ for $a \le t \le b$:

$$\text{Work} = \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$$

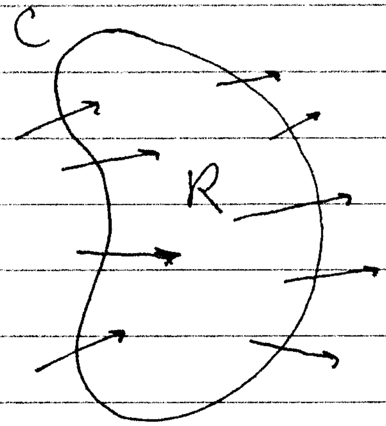
$$= \int_a^b \left(M \frac{dg}{dt} + N \frac{dh}{dt} + P \frac{dk}{dt} \right) dt$$

$$= \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$$

$$= \int_a^b \left(M\, dx + N\, dy + P\, dz \right)$$

Flux across a Closed Plane Curve

Consider region R inside a closed, stationary loop C sitting on the surface of a moving fluid. At each moment t some fluid enters R and some fluid exits R. We will assume that the fluid may compress (more dense) or expand (less dense) inside loop C.

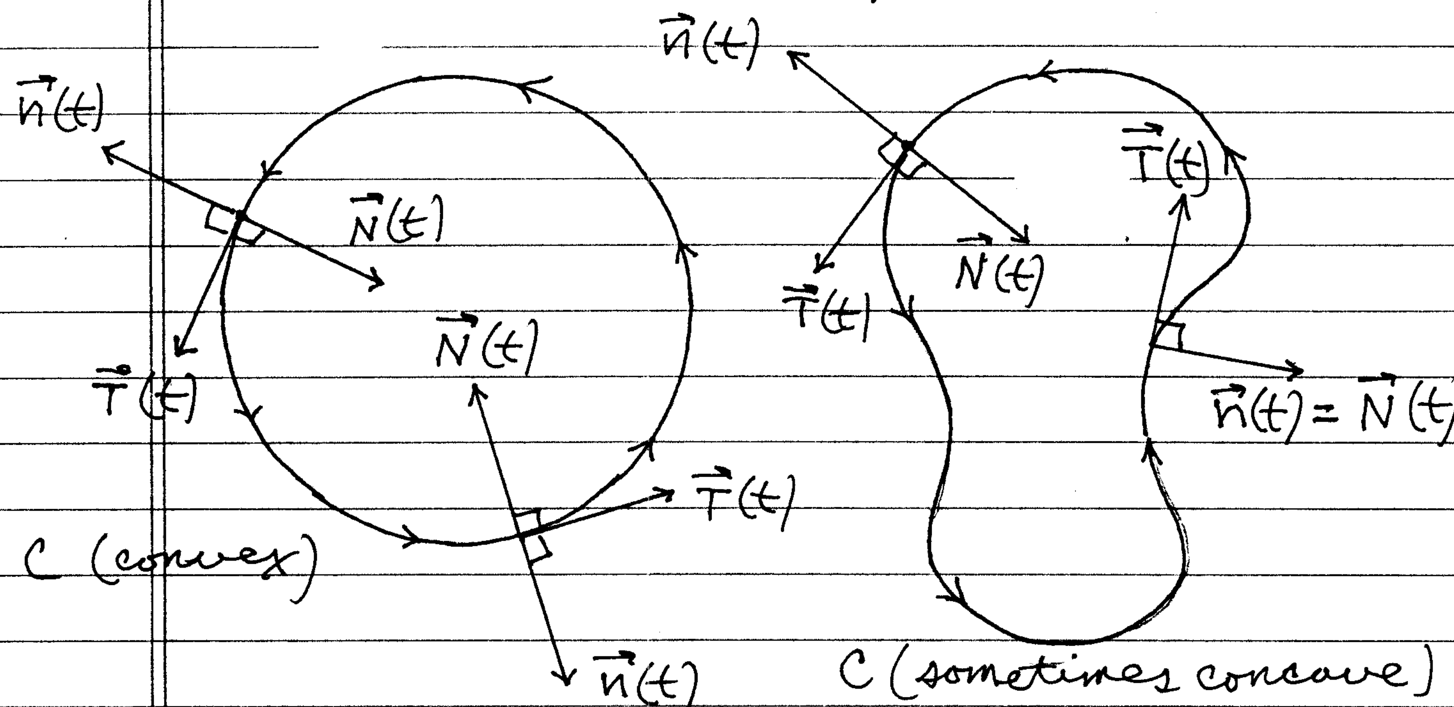


Question: Will the net flow rate of fluid across loop C be positive (more leaves than enters – expansion within R) or negative (more enters than leaves – compression within R) or zero?

The answer is called flux.

Recall: $\vec{N}(t)$ is the principal unit normal vector $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$,

where $\vec{T}(t)$ is the unit tangent vector.
Define vector $\vec{n}(t)$ to be the unit vector which is perpendicular to $\vec{T}(t)$ and pointing to the "exterior" of loop C. If C is always "convex", then $\vec{n}(t) = -\vec{N}(t)$. If C is sometimes "concave", then $\vec{n}(t) = \vec{N}(t)$ sometimes (See diagrams below.).



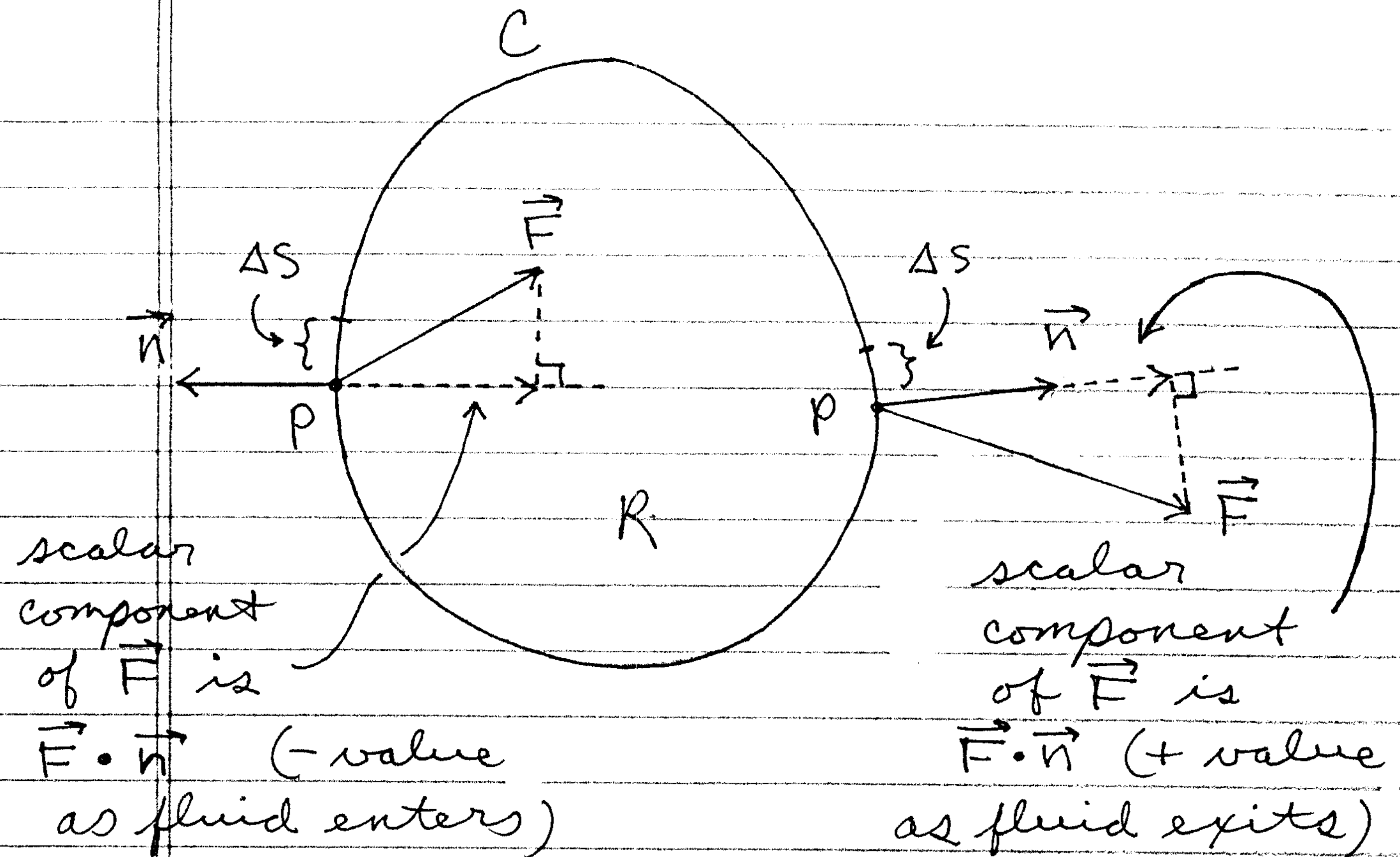
Call $\vec{n}(t)$ the <u>exterior</u> (unit) <u>normal vector</u>.

Assume that $\vec{F} = \delta \vec{v}$, where δ is the density of the fluid ($\frac{\text{mass}}{\text{area}}$ units) and $\vec{v}$ is the velocity vector for the moving fluid. The units for $\vec{F}$ will be

$$\frac{\text{mass}}{\text{area}} \times \frac{\text{length}}{\text{area}} = \frac{\text{mass}}{\text{length} \times \text{time}}.$$

The flow rate at point P on C along a small arc of length ΔS is approximately

$$(\vec{F} \cdot \vec{n})\,\Delta S \quad \left(\frac{\text{mass}}{\text{time}} \text{ units}\right).$$



__Def__: Let $\vec{F} = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ be a vector field defined on a smooth closed curve C in the plane. Let $\vec{n}$ be the outward-pointing unit normal vector on C. The __flux__ of $\vec{F}$ across C is

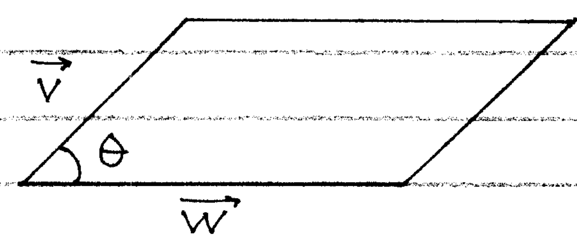
$$\text{Flux} = \int_C \vec{F}\cdot\vec{n}\; ds$$

__Recall__: If $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$ are vectors, then their __cross product__ is the vector given by

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= (v_2 w_3 - v_3 w_2)\,\vec{i} - (v_1 w_3 - v_3 w_1)\,\vec{j} + (v_1 w_2 - v_2 w_1)\,\vec{k} .$$

1.) $\vec{v} \times \vec{w}$ is $\perp$ to both $\vec{v}$ and $\vec{w}$.
2.) $\vec{v} \times \vec{w}$ is oriented according to the right-hand rule.
3.) The magnitude of $\vec{v} \times \vec{w}$ is $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$, the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$:



Method for Evaluating Flux

assume $C : \vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j}$ is oriented counter-clockwise.
Then $\boxed{\vec{n} = \vec{T} \times \vec{k}}$ (Show sketch.)

and $\vec{T}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{\frac{d}{dt}\vec{r}(t)}{\frac{ds}{dt}}$

$= \dfrac{d}{\cancel{dt}}\vec{r}(t) \cdot \dfrac{\cancel{dt}}{ds} = \dfrac{d}{ds}\vec{r}(t)$

$= \dfrac{d}{ds}\left(g(t)\vec{i} + h(t)\vec{j}\right)$

$= \dfrac{d}{ds}g(t)\vec{i} + \dfrac{d}{ds}h(t)\vec{j}$

$= \dfrac{dx}{ds}\vec{i} + \dfrac{dy}{ds}\vec{j}$, i.e.,

$$\vec{T}(t) = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j}$$; then

$\vec{n} = \vec{T} \times \vec{k}$

$= \left(\dfrac{dx}{ds}\vec{i} + \dfrac{dy}{ds}\vec{j}\right) \times \vec{k}$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \dfrac{dy}{ds}\vec{i} - \dfrac{dx}{ds}\vec{j}$, i.e.,

$$\boxed{\vec{n} = \frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j}}.$$

Thus,

$\vec{F} \cdot \vec{n} = \left(M\vec{i} + N\vec{j}\right) \cdot \left(\dfrac{dy}{ds}\vec{i} - \dfrac{dx}{ds}\vec{j}\right)$

$= M \cdot \dfrac{dy}{ds} - N \cdot \dfrac{dx}{ds} \longrightarrow$

$$\int_C \left(\vec{F} \cdot \vec{n}\right) ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds \rightarrow$$

$$\boxed{\text{Flux} = \oint_C M\,dy - N\,dx}$$