Section 16.2
Thomas Calculus
11th Ed.

Vector Fields and Flux Across
a Closed Path $C$ in the $xy$-Plane

(There are many different
Flux measures—magnetic,
mass, heat, energy,
momentum, etc. For the
sake of simplicity, I will
assume that Flux here is
related to fluid flow only.)

What is the meaning of Flux?

Recall that the Flow of velocity
vector field $\mathbf{F}$ is a measure
of fluid flow ALONG a path $C$
and is given by

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds,$$

where $\mathbf{T}$ is the
unit tangent vector.
In simple terms, the FLUX of velocity vector field $\mathbf{F}$ is a measure of fluid flow ACROSS a path $C$ enclosing a region $R$ in the $xy$-plane.

Now let's bring in precise mathematical definitions and computations.

Assume that $\mathbf{F}(x,y) = M(x,y) \mathbf{i} + N(x,y) \mathbf{j}$ is a velocity vector field for a fluid with density units "mass" and velocity units "length/time" area.

Consider region $R$ inside a closed path (loop) $C$ in the $xy$-plane sitting on top of this moving fluid.

At any moment in time and at any point on path $C$: 
1.) Some fluid ENTERS region $R$ across path $C$, \\
2.) Some fluid EXITS region $R$ across path $C$, \\
or 3.) no fluid enters or exits region $R$ across path $C$. \\

We will use a unit vector, call it $\mathbf{n}$, an **Outward-Pointing Unit Normal Vector**. Here are the properties of $\mathbf{n}$: \\

1.) $\mathbf{n}$ is a unit vector. \\
2.) $\mathbf{n}$ is $\perp$ to path $C$, i.e., $\mathbf{n}$ is $\perp$ to $\mathbf{T}$, the unit tangent vector to path $C$. \\
3.) $\mathbf{n}$ points outward from region $R$. 
Before we determine a formula for \( \hat{n} \), let's not confuse \( \hat{n} \) for \( \vec{N} \) the Principal Unit Normal Vector to path \( C \), which is also a unit vector, which is also \( \perp \) to path \( C \), but which points in the direction that path \( C \) is turning. Conclusion: Sometimes \( \hat{n} = \vec{N} \). Sometimes \( \hat{n} = -\vec{N} \). (See diagram below.)

If \( C \) is always "convex," then \( \hat{n}(t) = -\vec{N}(t) \). If \( C \) is sometimes "concave," then \( \hat{n}(t) = \vec{N}(t) \) sometimes. (See diagrams below.)
Recall:  

I. \[ \cos \theta = \frac{\overrightarrow{A} \cdot \overrightarrow{B}}{|\overrightarrow{A}| \cdot |\overrightarrow{B}|} \]

(Follows from Law of Cosines)

\[ \overrightarrow{proj}_B \overrightarrow{A} = (|\overrightarrow{A}| \cos \theta) \frac{\overrightarrow{B}}{|\overrightarrow{B}|} \]

\[ = \frac{|\overrightarrow{A}| (\overrightarrow{A} \cdot \overrightarrow{B})}{|\overrightarrow{A}| \cdot |\overrightarrow{B}|} \cdot \frac{\overrightarrow{B}}{|\overrightarrow{B}|} = \frac{(\overrightarrow{A} \cdot \overrightarrow{B}) \overrightarrow{B}}{|\overrightarrow{B}|^2} \]

we assume \( \overrightarrow{B} \) is a unit vector, then \( |\overrightarrow{B}| = 1 \), and

\[ \overrightarrow{proj}_B \overrightarrow{A} = (\overrightarrow{A} \cdot \overrightarrow{B}) \overrightarrow{B} \]

III. a.) If \( 0 \leq \theta < \frac{\pi}{2} \), then \( \cos \theta \) is (+), so that \((\overrightarrow{A} \cdot \overrightarrow{B})\) is (+).

b.) If \( \frac{\pi}{2} < \theta \leq \pi \), then \( \cos \theta \) is (-), so that \((\overrightarrow{A} \cdot \overrightarrow{B})\) is (-).
c.) If \( \theta = \frac{\pi}{2} \), then \( \cos \theta = 0 \)

and \( (A \cdot B) = 0 \).

Now here is how we will use \( \vec{n} \) to distinguish fluid flowing \textbf{IN} to region \( R \) from fluid flowing \textbf{OUT} of region \( R \).

\[ \vec{F} \cdot \vec{n} \text{ is } (-) \quad \text{ (fluid enters } R) \]

\[ \vec{F} \cdot \vec{n} \text{ is } (+) \quad \text{ (fluid exits } R) \]
Now consider a point on path \( C \), a small piece of arc length \( ds \), and

\[
\text{proj} \vec{F} = (\vec{F} \cdot \vec{n}) \vec{n}.
\]

The flow of fluid \textit{ACROSS} path \( C \) along the piece of arc length \( ds \) is approximately

\[
(\vec{F} \cdot \vec{n}) \ ds
\]

\[\uparrow\quad \uparrow\]
units: \( \left( \frac{\text{mass} \times \text{length}}{\text{area} \times \text{time}} \right) \times \text{length} = \frac{\text{mass}}{\text{time}} \).

Thus, the \textit{Total Flux} across path \( C \) is

\[
\text{Flux} = \oint_C (\vec{F} \cdot \vec{n}) \ ds
\]
Note: If total flux is

1.) **POSITIVE**, then more fluid exits region \( R \) than enters region \( R \), i.e., the fluid "expands" inside region \( R \).

2.) **NEGATIVE** then more fluid enters region \( R \) than exits region \( R \), i.e., the fluid "compresses" inside region \( R \).

3.) **ZERO**, then the same amount of fluid enters and exits region \( R \).

But we still don't have a formula for \( \overrightarrow{\mathbf{n}} \)! The next few pages will do this.

Recall: If \( \overrightarrow{\mathbf{v}} = v_1 \overrightarrow{i} + v_2 \overrightarrow{j} + v_3 \overrightarrow{k} \) and \( \overrightarrow{\mathbf{w}} = w_1 \overrightarrow{i} + w_2 \overrightarrow{j} + w_3 \overrightarrow{k} \) are vectors, then their **cross product is** the vector given by
\[ \vec{V} \times \vec{W} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2w_3 - v_3w_2) \hat{i} - (v_1w_3 - v_3w_1) \hat{j} + (v_1w_2 - v_2w_1) \hat{k} \]

1. \( \vec{V} \times \vec{W} \) is \( \perp \) to both \( \vec{V} \) and \( \vec{W} \).
2. \( \vec{V} \times \vec{W} \) is oriented according to the right-hand rule.
3. The magnitude of \( \vec{V} \times \vec{W} \) is \( |\vec{V} \times \vec{W}| = |\vec{V}| |\vec{W}| \sin \theta \), the area of the parallelogram formed by \( \vec{V} \) and \( \vec{W} \):

\[
\begin{array}{c}
\vec{V} \\
\downarrow \theta \\
\vec{W}
\end{array}
\]

Method for Evaluating Flux

Assume \( \vec{C} : \vec{r}(t) = g(t) \hat{i} + h(t) \hat{j} \)

is oriented counter-clockwise.

Then \( \vec{n} = \vec{r} \times \vec{k} \).
\[ \vec{n} = \vec{T} \times \vec{k} \]

(Use right-hand rule for Cross Product.)
\[ \text{and } \vec{T}(t) = \frac{\vec{r}(t)}{\|\vec{r}(t)\|} = \frac{\frac{d\vec{r}}{dt}}{\frac{d\vec{r}}{ds}} \]
\[ = \frac{d}{dt} \vec{r}(t) \cdot \frac{ds}{ds} = \frac{d}{ds} \vec{r}(t) \]
\[ = \frac{d}{ds} \left( g(t) \hat{i} + h(t) \hat{j} \right) \]
\[ = \frac{d}{ds} g(t) \hat{i} + \frac{d}{ds} h(t) \hat{j} \]
\[ = \frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j}, \text{ i.e.,} \]
\[ \vec{T}(t) = \frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j}; \text{ then} \]
\[ \vec{n} = \vec{T} \times \vec{k} \]
\[ = \left( \frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j} \right) \times \vec{k} \]
\[ = \begin{vmatrix} \hat{i} & \hat{j} & \vec{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{dy}{ds} \hat{i} - \frac{dx}{ds} \hat{j}, \text{ i.e.,} \]
\[ n = \frac{dy}{ds} \hat{i} - \frac{dx}{ds} \hat{j}. \]

Thus,
\[ \vec{F} \cdot \vec{n} = (M \hat{i} + N \hat{j}) \cdot \left( \frac{dy}{ds} \hat{i} - \frac{dx}{ds} \hat{j} \right) \]
\[ = M \cdot \frac{dy}{ds} - N \cdot \frac{dx}{ds} \rightarrow \]
\[ \oint \vec{F} \cdot \vec{n} \, ds = \oint (M \frac{dy}{ds} - N \frac{dx}{ds}) \, ds \rightarrow \]
Flux = \int_{c} M \, dy - N \, dx

= \int_{a}^{b} \left( M \frac{dy}{dt} - N \frac{dx}{dt} \right) \, dt