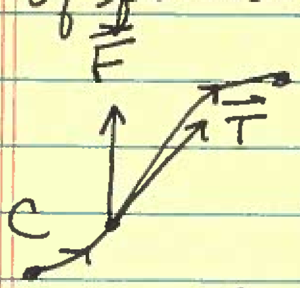
Section 16.2
Thomas Calculus
11th Ed.

Vector Fields and Flux Across a Closed Path C in the XY-Plane

(There are many different Flux measures – magnetic, mass, heat, energy, momentum, etc. For the sake of simplicity, I will assume that Flux here is related to fluid flow only.)

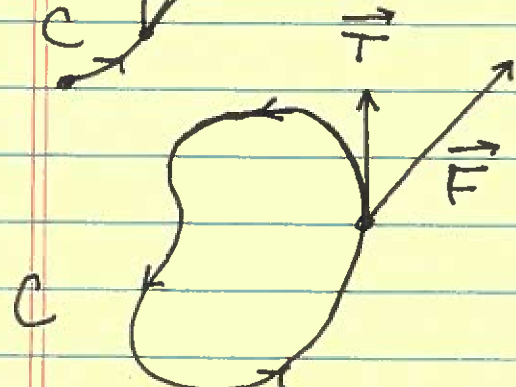
What is the meaning of Flux?

Recall that the FLOW of velocity vector field $\vec{F}$ is a measure of fluid flow ALONG a path C



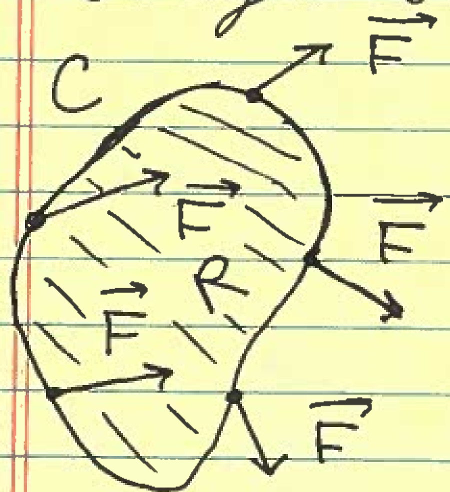
and is given by

$$\text{Flow} = \int_C \vec{F} \cdot \vec{T}\, ds,$$



where $\vec{T}$ is the unit tangent vector.

In simple terms, the FLUX of velocity vector field $\vec{F}$ is a measure of fluid flow ACROSS a path C enclosing a region R in the XY-plane.



Now let's bring in precise mathematical definitions and computations.

Assume that $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ is a velocity vector field for a fluid with density units "$\frac{\text{mass}}{\text{area}}$" and velocity units "$\frac{\text{length}}{\text{time}}$"

Consider region R inside a closed path (loop) C in the XY-plane sitting on top of this moving fluid.

At any moment in time and at any point on path C:

1.) Some fluid ENTERS region R across path C,

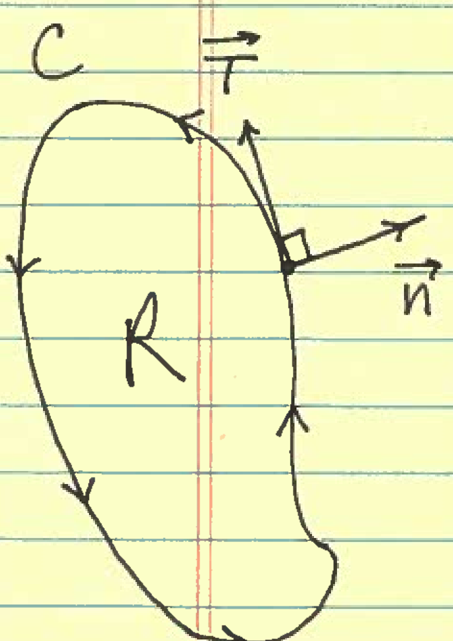
2.) Some fluid EXITS region R across path C,

or 3.) no fluid enters or exits region R across path C.

We will use a unit vector, call it $\vec{n}$, an <u>Outward-Pointing</u> <u>Unit Normal</u> <u>Vector</u>. Here are the properties of $\vec{n}$:

1.) $\vec{n}$ is a unit vector.

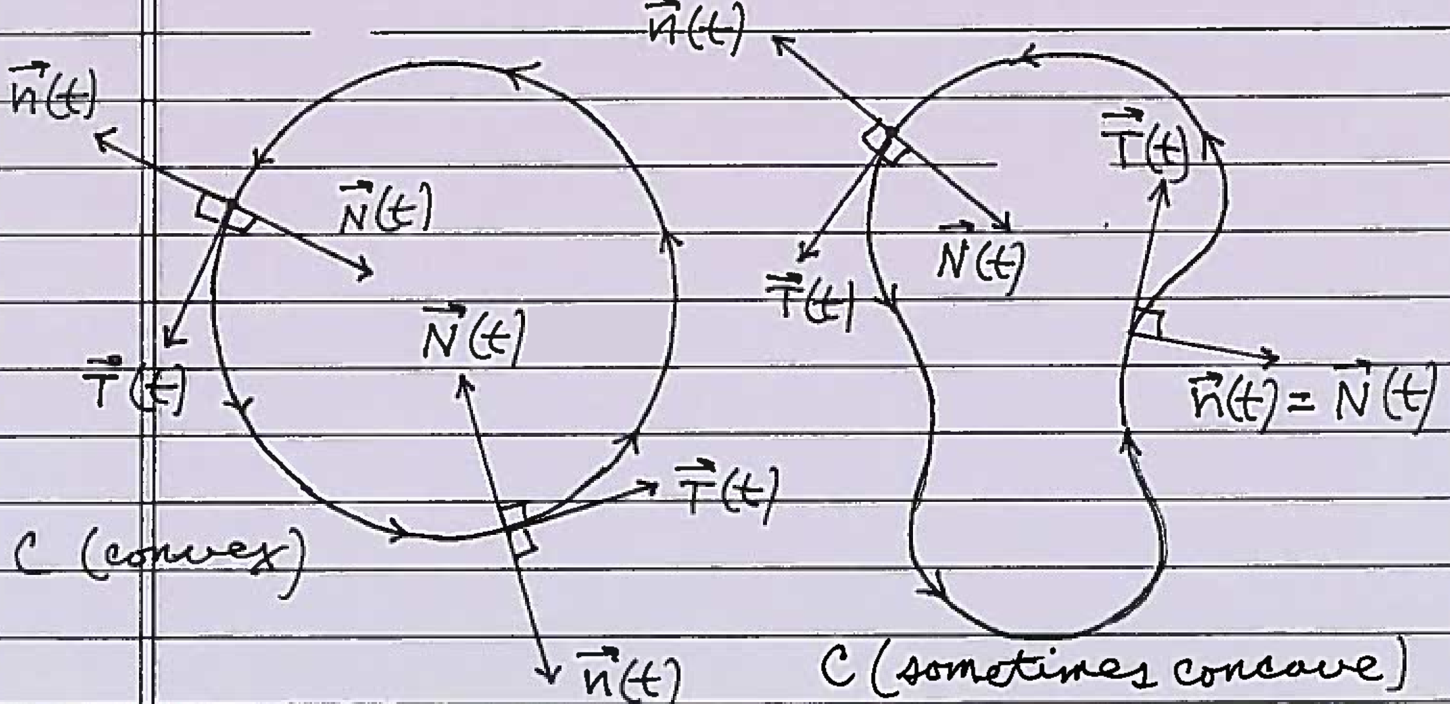
2.) $\vec{n}$ is $\perp$ to path C, i.e., $\vec{n}$ is $\perp$ to $\vec{T}$, the unit tangent vector to path C.



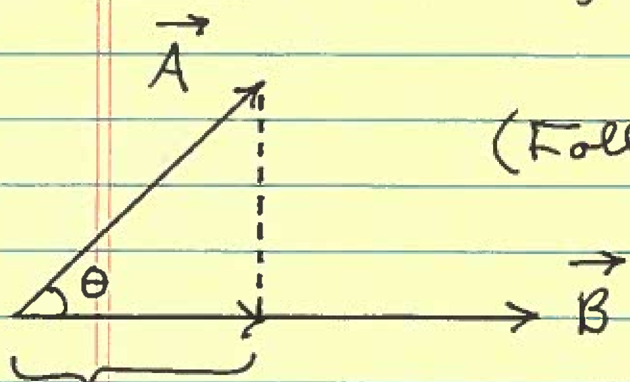
3.) $\vec{n}$ points outward from region R.

Before we determine a formula for $\vec{n}$, let's not confuse $\vec{n}$ for $\vec{N}$, the Principal Unit Normal Vector to path C, which is also a unit vector, which is also ⊥ to path C, BUT which points in the direction that path C is turning. Conclusion: Sometimes $\vec{n} = \vec{N}$. Sometimes $\vec{n} = -\vec{N}$. (See diagram below)

If C is always "convex", then $\vec{n}(t) = -\vec{N}(t)$. If C is sometimes "concave", then $\vec{n}(t) = \vec{N}(t)$ sometimes (See diagrams below.).



Recall: I.) $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|\vec{A}|\,|\vec{B}|}$



$\text{proj}_{\vec{B}}\vec{A}$

(Follows from Law of Cosines.)

II.) $\text{proj}_{\vec{B}}\vec{A} = (|\vec{A}|\cos\theta)\,\dfrac{\vec{B}}{|\vec{B}|}$

$$= \cancel{|\vec{A}|}\cdot\frac{(\vec{A}\cdot\vec{B})}{\cancel{|\vec{A}|}\,|\vec{B}|}\cdot\frac{\vec{B}}{|\vec{B}|} = \frac{(\vec{A}\cdot\vec{B})\,\vec{B}}{|\vec{B}|^2}$$

; if we assume $\vec{B}$ is a <u>unit vector</u>, then $|\vec{B}| = 1$, and

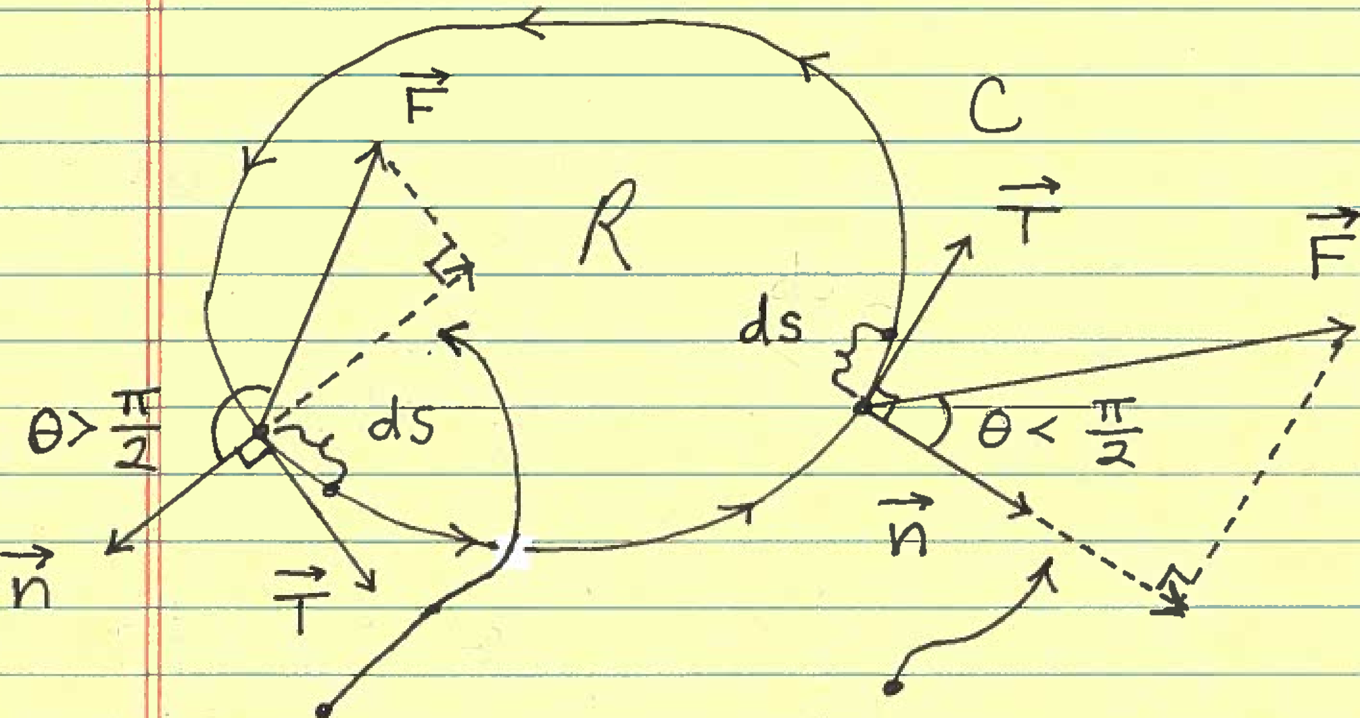
$$\boxed{\text{proj}_{\vec{B}}\vec{A} = (\vec{A}\cdot\vec{B})\,\vec{B}}$$

III.) a.) If $0 \le \theta < \frac{\pi}{2}$, then $\cos\theta$ is (+), so that $(\vec{A}\cdot\vec{B})$ is (+).

b.) If $\frac{\pi}{2} < \theta \le \pi$, then $\cos\theta$ is (−), so that $(\vec{A}\cdot\vec{B})$ is (−).

c.) If $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$
and $(A \cdot B) = 0$.

Now here is how we will use $\vec{n}$ to distinguish fluid flowing IN to region R from fluid flowing OUT of region R.



$\vec{F} \cdot \vec{n}$ is (−)
(fluid enters R)

$\vec{F} \cdot \vec{n}$ is (+)
(fluid exits R)

Now consider a point on path C, a small piece of arc length ds, and

$$\text{proj}_{\vec{n}} \vec{F} = (\vec{F} \cdot \vec{n})\, \vec{n} .$$

The flow of fluid ACROSS path C along the piece of arc length ds is approximately

$$(\vec{F} \cdot \vec{n}) \quad ds$$

units: $\left(\frac{\text{mass}}{\text{area}} \times \frac{\text{length}}{\text{time}}\right) \times (\text{length}) = \frac{\text{mass}}{\text{time}}$.

Thus, the Total Flux across path C is

$$\boxed{\text{Flux} = \oint_C (\vec{F} \cdot \vec{n})\, ds}$$

Note: If Total Flux is

1.) POSITIVE, then more fluid exits region R than enters region R, i.e., the fluid "expands" inside region R.

2.) NEGATIVE then more fluid enters region R than exits region R, i.e., the fluid "compresses" inside region R.

3.) ZERO, then the same amount of fluid enters and exits region R.

But we still don't have a formula for $\vec{n}$! The next few pages will do this.

Recall: If $\vec{V} = V_1 \vec{i} + V_2 \vec{j} + V_3 \vec{K}$ and $\vec{w} = w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{K}$ are vectors, then their cross product is the vector given by

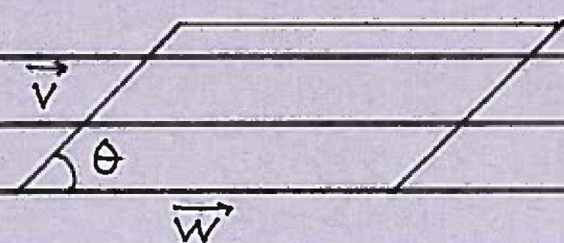
$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= (v_2 w_3 - v_3 w_2)\,\vec{i} - (v_1 w_3 - v_3 w_1)\,\vec{j} + (v_1 w_2 - v_2 w_1)\,\vec{k}.$$

1.) $\vec{v} \times \vec{w}$ is $\perp$ to both $\vec{v}$ and $\vec{w}$.

2.) $\vec{v} \times \vec{w}$ is oriented according to the right-hand rule.

3.) The magnitude of $\vec{v} \times \vec{w}$ is $|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin\theta$, the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$:



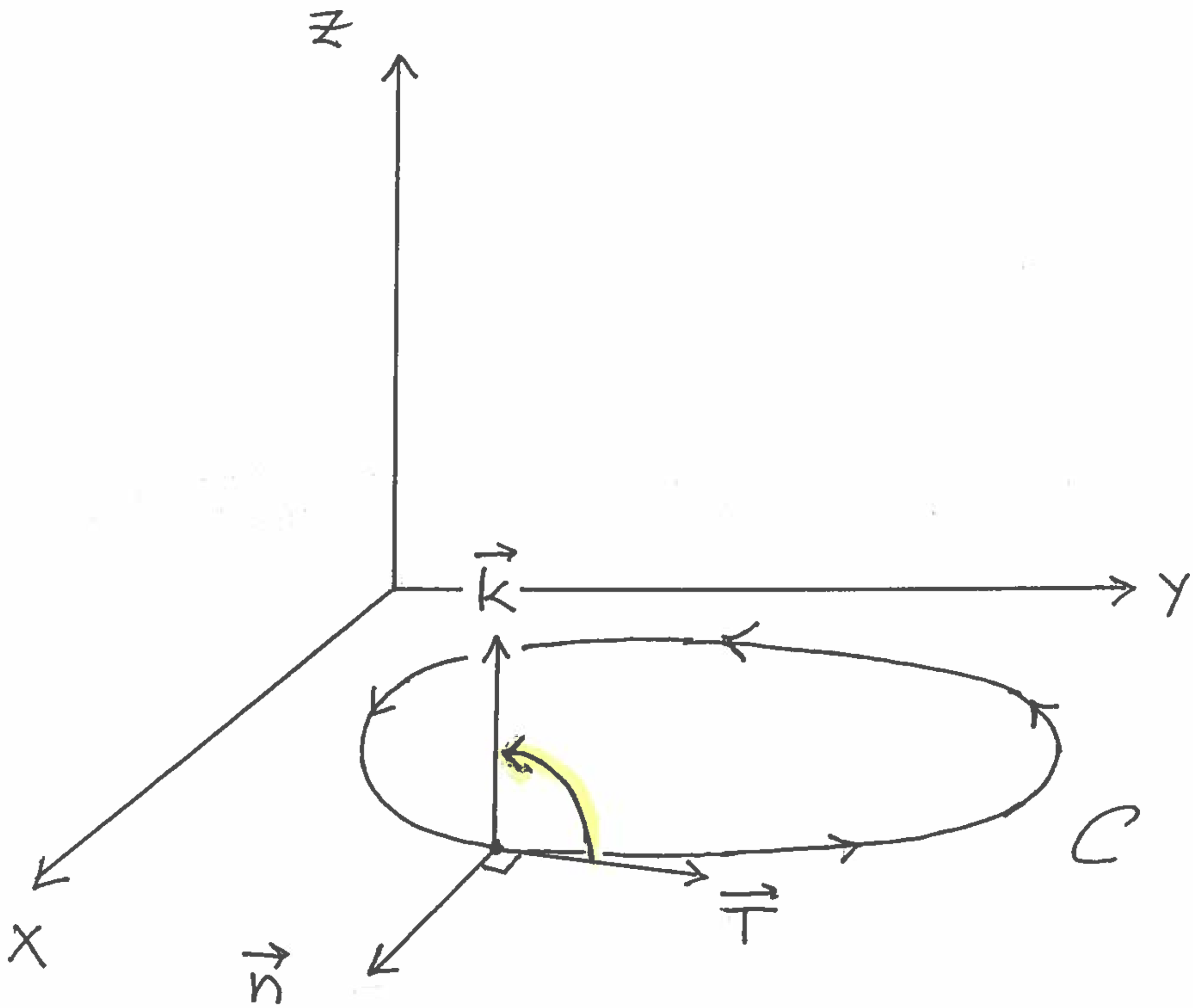
<u>Method for Evaluating Flux</u>

Assume $C: \vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j}$ is oriented counter-clockwise.

Then $\boxed{\vec{n} = \vec{T} \times \vec{k}}$

$$\boxed{\vec{n} = \vec{T} \times \vec{k}}$$

(Use right-hand rule for Cross Product.)



and $\vec{T}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{\frac{d}{dt}\vec{r}(t)}{\frac{ds}{dt}}$

$= \dfrac{d}{\cancel{dt}}\vec{r}(t) \cdot \dfrac{\cancel{dt}}{ds} = \dfrac{d}{ds}\vec{r}(t)$

$= \dfrac{d}{ds}\left(g(t)\vec{i} + h(t)\vec{j}\right)$

$= \dfrac{d}{ds}g(t)\vec{i} + \dfrac{d}{ds}h(t)\vec{j}$

$= \dfrac{dx}{ds}\vec{i} + \dfrac{dy}{ds}\vec{j}$, i.e.,

$\vec{T}(t) = \dfrac{dx}{ds}\vec{i} + \dfrac{dy}{ds}\vec{j}$; then

$\vec{n} = \vec{T} \times \vec{k}$

$= \left(\dfrac{dx}{ds}\vec{i} + \dfrac{dy}{ds}\vec{j}\right) \times \vec{k}$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \dfrac{dy}{ds}\vec{i} - \dfrac{dx}{ds}\vec{j}$, i.e.,

$\boxed{\vec{n} = \dfrac{dy}{ds}\vec{i} - \dfrac{dx}{ds}\vec{j}}$.

Thus,

$\vec{F} \cdot \vec{n} = (M\vec{i} + N\vec{j}) \cdot \left(\dfrac{dy}{ds}\vec{i} - \dfrac{dx}{ds}\vec{j}\right)$

$= M \cdot \dfrac{dy}{ds} - N \cdot \dfrac{dx}{ds} \longrightarrow$

$\displaystyle\int_C (\vec{F} \cdot \vec{n})\, ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds \rightarrow$

$$\text{Flux} = \oint_C M\,dy - N\,dx$$

$$= \int_a^b \left(M \frac{dy}{dt} - N \frac{dx}{dt} \right) dt$$