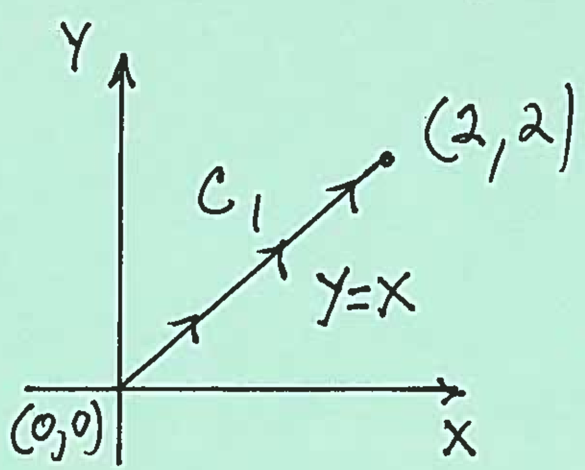
Section 16.3
Thomas Calculus
11th Ed.

Work Done by Gradient Vector Fields

Example A: Consider the scalar function $f(x,y) = x^2 y^3$ and its Gradient Vector Field

$$\vec{\nabla} f = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}$$
$$= (2xy^3)\,\vec{i} + (3x^2y^2)\,\vec{j} \,.$$

1.) Compute the work done by $\vec{F} = \vec{\nabla} f$ along path C_1 from $(0,0)$ to $(2,2)$ given by



C_1: $x = t$, $y = t$ for $0 \le t \le 2$;

$$\text{Work} = \int_{C_1} \vec{F} \cdot \vec{T}\, ds$$

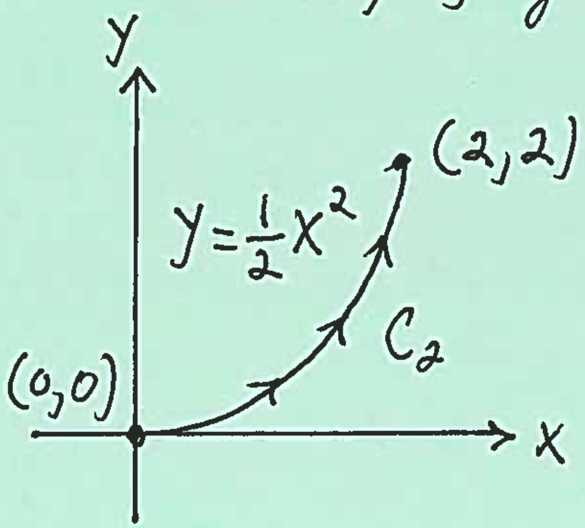
$$= \int_{C_1} M\,dx + N\,dy = \int_0^2 \left[M \cdot \frac{dx}{dt} + N \cdot \frac{dy}{dt}\right] dt$$

$$= \int_0^2 \left[(2xy^3) \cdot \frac{dx}{dt} + (3x^2y^2)\frac{dy}{dt}\right] dt$$

$$= \int_0^2 \left[2(t)(t^3)(1) + 3(t^2)(t^2)(1)\right] dt$$

$$= \int_0^2 5t^4\,dt = t^5 \Big|_0^2 = \boxed{32} .$$

2.) Compute the work done by $\vec{F} = \vec{\nabla} f$ along path C_2 from $(0,0)$ to $(2,2)$ given by



C_2: $x = t$, $y = \frac{1}{2}t^2$ for $0 \le t \le 2$;

$$\text{Work} = \int_{C_2} \vec{F} \cdot \vec{T}\,ds$$

$$= \int_{C_2} M\,dx + N\,dy = \int_0^2 \left[(2xy^3) \cdot \frac{dx}{dt} + (3x^2y^2) \cdot \frac{dy}{dt}\right] dt$$

$$= \int_0^2 \left[2(t)\left(\tfrac{1}{2}t^2\right)^3(1) + 3(t^2)\left(\tfrac{1}{2}t^2\right)^2\cdot(t) \right] dt$$

$$= \int_0^2 \left[\tfrac{1}{4}t^7 + \tfrac{3}{4}t^7 \right] dt = \int_0^2 t^7\, dt$$

$$= \frac{1}{8}t^8 \Big|_0^2 = 2^5 = \boxed{32} \;!$$

It is NOT a coincidence that the answers are the same!

<u>Example B</u>: Consider the scalar function $f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$ and its Gradient Vector Field

$$\vec{\nabla} f = f_x(x,y,z)\,\vec{i} + f_y(x,y,z)\,\vec{j} + f_z(x,y,z)\,\vec{k}$$

$$= x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

1.) Compute the Work done by $\vec{F} = \vec{\nabla} f$ along path C_1 from $(0,0,0)$ to $(2,-1,1)$ given by

$$C_1 : \vec{r}_1(t) = (2t)\,\vec{i} + (-t^2)\,\vec{j} + (t^3)\,\vec{k}$$

for $0 \le t \le 1$; Work $= \int_{C_1} \vec{F}\cdot\vec{T}\, ds$

$$= \int_{C_1} M\,dx + N\,dy + P\,dz$$

$$= \int_{C_1} \left[(x)\cdot\frac{dx}{dt} + (y)\cdot\frac{dy}{dt} + (z)\cdot\frac{dz}{dt} \right] dt$$

$$= \int_0^1 \left[(2t)(2) + (-t^2)(-2t) + (t^3)(3t^2) \right] dt$$

$$= \int_0^1 \left[4t + 2t^3 + 3t^5 \right] dt$$

$$= \left(2t^2 + \frac{1}{2}t^4 + \frac{1}{2}t^6 \right) \Big|_0^1$$

$$= 2 + \frac{1}{2} + \frac{1}{2} = \boxed{3} \, .$$

2.) Compute the work done by $\vec{F} = \vec{\nabla} f$ along path C_2 from $(0,0,0)$ to $(2,-1,1)$ given by

C_2: $\vec{r}_2(t) = (2\sqrt{t})\vec{i} + (-t)\vec{j} + (t^4)\vec{k}$

for $0 \le t \le 1$; Work $= \int_{C_2} \vec{F} \bullet \vec{T}\, ds$

$$= \int_{C_2} M\,dx + N\,dy + P\,dz$$

$$= \int_0^1 \left[(X) \cdot \frac{dX}{dt} + (Y) \cdot \frac{dY}{dt} + (Z) \cdot \frac{dZ}{dt} \right] dt$$

$$= \int_0^1 \left[(2\sqrt{t}) \cdot \frac{1}{\sqrt{t}} + (-t)(-1) + (t^4) \cdot (4t^3) \right] dt$$

$$= \int_0^1 \left[2 + t + 4t^7 \right] dt$$

$$= \left(2t + \frac{1}{2}t^2 + \frac{1}{2}t^8 \right) \Big|_0^1$$

$$= 2 + \frac{1}{2} + \frac{1}{2} = \boxed{3} \ !!$$

It is not a coincidence that the answers are the same.

The next few pages of notes will explain why it happens.

But first, here is another cool aspect of Gradient Vector Fields.

Watch this mathematical magic!

Note 1: In Example A, we have $f(x,y) = x^2y^3$ with Gradient Vector Field

$$\vec{F} = \vec{\nabla} f = (2xy^3)\vec{i} + (3x^2y^2)\vec{j}.$$

In 1.) $C_1: x = t,\ y = t$ for $0 \le t \le 2$ is a path from $(0,0)$ to $(2,2)$;

$$\text{Work} = \int_{C_1} \vec{F} \cdot \vec{T}\, ds = \boxed{32} \quad ; \text{ and}$$

$$f(x,y)\Big|_{(0,0)}^{(2,2)} = f(2,2) - f(0,0)$$

$$= 2^2 2^3 - 0^2 0^3 = \boxed{32} \ \ldots \text{ hmmm!?}$$

Note 2: In Example B, we have $f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$ with Gradient Vector Field

$$\vec{F} = \vec{\nabla} f = x\vec{i} + y\vec{j} + z\vec{k}.$$

In 1.) $C_1: x = 2t,\ y = -t^2,\ z = t^3$ for $0 \le t \le 1$ is a path from $(0,0,0)$

to $(2,-1,1)$;

$$\text{Work} = \int_{C_1} \vec{F} \cdot \vec{T}\, ds = \boxed{3} \;;\; \text{and}$$

$$f(x,y,z)\Big|_{(0,0,0)}^{(2,-1,1)} = f(2,-1,1) - f(0,0,0)$$

$$= \left(\tfrac{1}{2}(2)^2 + \tfrac{1}{2}(-1)^2 + \tfrac{1}{2}(1)^2\right)$$

$$- \left(\tfrac{1}{2}(0)^2 + \tfrac{1}{2}(0)^2 + \tfrac{1}{2}(0)^2\right)$$

$$= 2 + \tfrac{1}{2} + \tfrac{1}{2} = \boxed{3} \;\ldots\; \text{Cool!}$$

Example: Find a scalar function f which has the given Gradient Vector Field.

1.) $\vec{\nabla} f = (2x+y)\vec{i} + (x+3)\vec{j}$; then

$$f_x = 2x+y \xrightarrow{\int_x} f = x^2 + xy + g(y)$$

$$\xrightarrow{D_y} f_y = 0 + x + g'(y) = x+3 \longrightarrow$$

$$g'(y) = 3 \rightarrow g(y) = 3y + \cancel{c}^{\,0}, \text{ so}$$

$$\boxed{f(x,y) = x^2 + xy + 3y}.$$

2.) $\vec{\nabla} f = \left(2x^2 y e^{x^2 y} + e^{x^2 y} + \dfrac{y^3}{x}\right)\vec{i}$

$$+ \left(x^3 e^{x^2 y} + 3y^2 \ln x\right)\vec{j}; \text{ then}$$

$$f_y = x^3 e^{x^2 y} + 3y^2 \ln x \xrightarrow{\int_y}$$

$$f = x^3 \cdot e^{x^2 y} \cdot \frac{1}{x^2} + y^3 \ln x + g(x)$$

$$= x e^{x^2 y} + y^3 \ln x + g(x) \xrightarrow{D_x}$$

$$f_x = x \cdot e^{x^2y} \cdot 2xy + (1)\, e^{x^2y} + y^3 \cdot \frac{1}{x} + g'(x)$$

$$= 2x^2ye^{x^2y} + e^{x^2y} + \frac{y^3}{x} + g'(x)$$

$$= 2x^2ye^{x^2y} + e^{x^2y} + \frac{y^3}{x} \longrightarrow$$

$g'(x) = 0 \longrightarrow g(x) = \cancel{c}^{\,0}$, ~~so~~

$$\boxed{f(x,y) = xe^{x^2y} + y^3 \ln x} .$$

3.) $\vec{\nabla} f = (2x)\,\vec{i} + (1 - ze^{-yz})\,\vec{j} + (-ye^{-yz} - 1)\,\vec{k}$;

then

$$f_x = 2x \xrightarrow{\int_x} f = x^2 + g(y,z) \xrightarrow{D_y}$$

$$f_y = 0 + g_y(y,z) = 1 - ze^{-yz} \xrightarrow{\int_y}$$

$$g(y,z) = y - ze^{-yz} \cdot \frac{1}{-z} + k(z)$$

$= y + e^{-yz} + k(z)$, then

$$f = x^2 + y + e^{-yz} + k(z) \xrightarrow{D_z}$$

$$f_z = 0 + 0 - ye^{-yz} + k'(z)$$

$$= -ye^{-yz} - 1 \rightarrow k'(z) = -1 \xrightarrow{\int}$$

$$k(z) = -z + \cancel{C}^{\,0} \rightarrow$$

$$\boxed{f(x,y,z) = x^3 + y + e^{-yz} - z}$$

4.) $\vec{\nabla} f = (2xz - yz\sin(xyz) + 1)\,\vec{i}$

$+ (z^3 - xz\sin(xyz) - 2)\,\vec{j}$

$+ (x^2 + 3yz^2 - xy\sin(xyz) + 3)\,\vec{k}$;

then

$$f_x = 2xz - yz\sin(xyz) + 1 \xrightarrow{\int_x}$$

$$f = x^2 z - yz \cdot -\cos(xyz) \cdot \frac{1}{yz} + x + g(y,z)$$

$$= x^2 z + \cos(xyz) + x + g(y,z) \xrightarrow{D_y}$$

$$f_y = 0 - xz\sin(xyz) + 0 + g_y(y,z)$$

$$= z^3 - xz\sin(xyz) - 2 \rightarrow$$

$$g_y(y,z) = z^3 - 2 \xrightarrow{\int_y}$$

$$g(y,z) = yz^3 - 2y + k(z) \longrightarrow$$

$$f = x^2 z + \cos(xyz) + x + yz^3 - 2y + k(z) \xrightarrow{D_z}$$

$$f_z = x^2 - xy\sin(xyz) + 0 + 3yz^2 - 0 + k'(z)$$

$$= x^2 + 3yz^2 - xy\sin(xyz) + 3 \longrightarrow$$

$$k'(z) = 3 \xrightarrow{\int} k(z) = 3z + \cancel{C}^{0}, \text{ so}$$

$$\boxed{f(x,y,z) = x^2 z + \cos(xyz) + x + yz^3 - 2y + 3z}$$