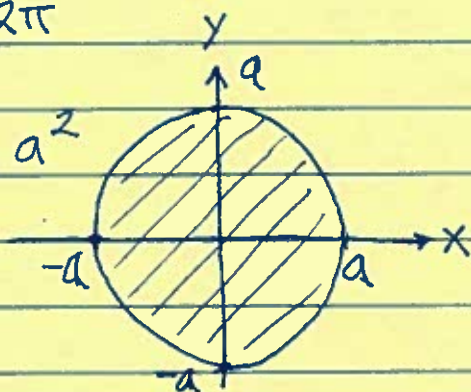


## Section 16.4

$$C: \begin{cases} x = a \cos t \\ y = a \sin t \end{cases} \text{ for } 0 \leq t \leq 2\pi$$

$$x^2 + y^2 \leq a^2$$



$$\frac{dx}{dt} = -a \sin t,$$

$$\frac{dy}{dt} = a \cos t;$$

1.)  $\vec{F}(x, y) = (-y)\vec{i} + (x)\vec{j}$

a.)  $\oint_C \vec{F} \cdot \vec{n} \, ds = \int_C M \, dy - N \, dx$

$$= \int_0^{2\pi} [(-y) \frac{dy}{dt} - (x) \frac{dx}{dt}] \, dt$$

$$= \int_0^{2\pi} [(-a \sin t)(a \cos t) - (a \cos t)(-a \sin t)] \, dt$$

$$= \int_0^{2\pi} 0 \, dt = 0;$$

$$\iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA = \iint_R (0 + 0) dA = 0$$

b.)  $\oint_C \vec{F} \cdot \vec{T} \, ds = \int_C M \, dx + N \, dy$

$$= \int_0^{2\pi} [(-y) \frac{dx}{dt} + (x) \frac{dy}{dt}] \, dt$$

$$= \int_0^{2\pi} [(-a \sin t)(-a \sin t) + (a \cos t)(a \cos t)] \, dt$$

$$= \int_0^{2\pi} [a^2 (\sin^2 t + \cos^2 t)] \, dt$$

$$= a^2 t \Big|_0^{2\pi} = 2a^2 \pi;$$

$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_R (1 - (-1)) dA$$

$$= \iint_R 2 dA = 2 \cdot \iint_R 1 dA = 2(\text{area } R)$$

$$= 2(\pi a^2) = 2a^2\pi$$

$$2.) \vec{F}(x, y) = (y)\vec{i} + (0)\vec{j}$$

$$a.) \oint_C \vec{F} \cdot \vec{n} ds = \int_C M dy - N dx$$

$$= \int_0^{2\pi} \left[ (y) \frac{dy}{dt} - (0) \frac{dx}{dt} \right] dt$$

$$= \int_0^{2\pi} [(a \sin t)(a \cos t)] dt$$

$$= a^2 \cdot \frac{1}{2} \sin^2 t \Big|_0^{2\pi} = \frac{a^2}{2} \sin^2 2\pi - \frac{a^2}{2} \sin^2 0 = 0;$$

$$\iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA = \iint_R (0 + 0) dA = 0$$

$$b.) \oint_C \vec{F} \cdot \vec{T} ds = \int_C M dx + N dy$$

$$= \int_0^{2\pi} \left[ (y) \frac{dx}{dt} + (0) \frac{dy}{dt} \right] dt$$

$$= \int_0^{2\pi} (a \sin t)(-a \sin t) dt$$

$$= -a^2 \int_0^{2\pi} \frac{1}{2} (1 - \cos 2t) dt$$

$$= -\frac{a^2}{2} \left( t - \frac{1}{2} \sin 2t \right) \Big|_0^{2\pi}$$

$$= \frac{-a^2}{2} \left( 2\pi - \frac{1}{2} \sin^2 4\pi \right) - \frac{-a^2}{2} \left( 0 - \frac{1}{2} \sin^2 0 \right)$$

$$= -a^2 \pi$$

$$\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_R (0 - (-1)) dA$$

$$= - \iint_R 1 dA = -(\text{area } R) = -\pi a^2 = -a^2 \pi.$$

$$4.) \vec{F}(x, y) = (-x^2 y) \vec{i} + (xy^2) \vec{j}$$

$$a.) \oint_C \vec{F} \cdot \vec{n} ds = \int_C M dy - N dx$$

$$= \int_0^{2\pi} \left[ (-x^2 y) \frac{dy}{dt} - (xy^2) \frac{dx}{dt} \right] dt$$

$$= \int_0^{2\pi} \left[ -(a^2 \cos^2 t)(a \sin t) \cdot (a \cos t) \right. \\ \left. - (a \cos t)(a^2 \sin^2 t)(-a \sin t) \right] dt$$

$$= \int_0^{2\pi} \left[ -a^4 \cos^3 t \sin t + a^4 \sin^3 t \cos t \right] dt$$

$$= \left( -a^4 \cdot \frac{1}{4} \cos^4 t + a^4 \cdot \frac{1}{4} \sin^4 t \right) \Big|_0^{2\pi}$$

$$= \left( \frac{1}{4} a^4 \cos^4 2\pi + \frac{1}{4} a^4 \sin^4 2\pi \right) \\ - \left( \frac{1}{4} a^4 \cos^4 0 + \frac{1}{4} a^4 \sin^4 0 \right)$$

$$= \frac{1}{4} a^4 (1)^4 - \frac{1}{4} a^4 (1)^4 = 0 ;$$

$$\iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA = \iint_R (-2xy + 2xy) dA$$

$$= \iint_R 0 \, dA = 0$$

$$b.) \oint_C \vec{F} \cdot \vec{T} \, ds = \int_C M \, dx + N \, dy$$

$$= \int_0^{2\pi} [(-x^2 y) \frac{dx}{dt} + (xy^2) \frac{dy}{dt}] \, dt$$

$$= \int_0^{2\pi} [(-a^2 \cos^2 t)(a \sin t)(-a \sin t) + (a \cos t)(a^2 \sin^2 t)(a \cos t)] \, dt$$

$$= \int_0^{2\pi} [a^4 \cos^2 t \sin^2 t + a^4 \cos^2 t \sin^2 t] \, dt$$

$$= 2a^4 \int_0^{2\pi} [\cos t \sin t]^2 \, dt$$

$$= 2a^4 \int_0^{2\pi} \left[\frac{1}{2} \sin 2t\right]^2 \, dt$$

$$= 2a^4 \cdot \frac{1}{4} \int_0^{2\pi} \sin^2 2t \, dt$$

$$= \frac{1}{2} a^4 \int_0^{2\pi} \frac{1}{2} (1 - \cos 4t) \, dt$$

$$= \frac{1}{4} a^4 \left( t - \frac{1}{4} \sin 4t \right) \Big|_0^{2\pi}$$

$$= \frac{1}{4} a^4 (2\pi - \frac{1}{4} \sin 8\pi) - \frac{1}{4} a^4 (0 - \frac{1}{4} \sin 0)$$

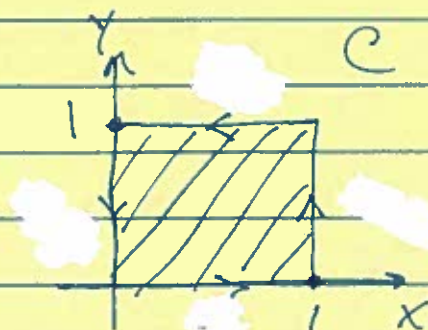
$$= \frac{1}{2} a^4 \pi \quad ;$$

$$\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA = \iint_R (y^2 - (-x^2)) \, dA$$

$$= \iint_R (a^2 \sin^2 t + a^2 \cos^2 t) \, dA$$

$$\begin{aligned}
 &= \iint_R a^2 (\underbrace{\sin^2 t + \cos^2 t}_1) dA \\
 &= a^2 \iint_R 1 dA = a^2 (\text{area } R) \\
 &= a^2 \cdot \pi (a^2) = a^4 \pi .
 \end{aligned}$$

5.)  $\vec{F}(x,y) = (x-y)\vec{i} + (y-x)\vec{j}$



a.) Circ =  $\oint_C \vec{F} \cdot \vec{T} ds$

$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R (-1 - (-1)) dA = \iint_R 0 dA = 0$$

b.) Flux =  $\oint_C \vec{F} \cdot \vec{n} ds = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$

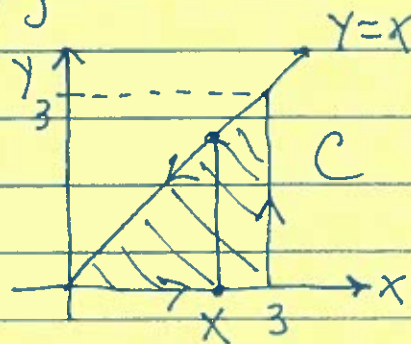
$$= \iint_R (1+1) dA = 2 \iint_R 1 dA = 2 (\text{area } R)$$

$$= 2(1) = 2$$

7.)  $\vec{F}(x,y) = (y^2 - x^2)\vec{i} + (x^2 + y^2)\vec{j}$

a.) Circ =  $\oint_C \vec{F} \cdot \vec{T} ds$

$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$



$$\begin{aligned}
&= \iint_R (2x - 2y) dA = \int_0^3 \int_0^x (2x - 2y) dy dx \\
&= \int_0^3 (2xy - y^2) \Big|_{y=0}^{y=x} dx \\
&= \int_0^3 (2x^2 - x^2) dx = \int_0^3 x^2 dx \\
&= \frac{1}{3} x^3 \Big|_0^3 = 9
\end{aligned}$$

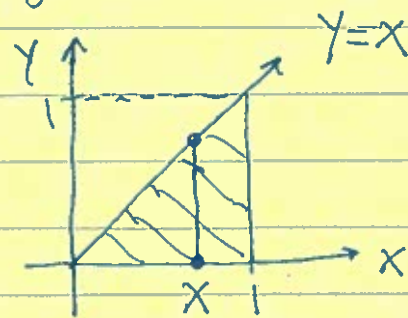
$$\begin{aligned}
\text{b.) Flux} &= \oint_C \vec{F} \cdot \vec{n} ds = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA \\
&= \iint_R (-2x + 2y) dA = \int_0^3 \int_0^x (2y - 2x) dy dx \\
&= \int_0^3 (y^2 - 2xy) \Big|_{y=0}^{y=x} dx \\
&= \int_0^3 (x^2 - 2x^2) dx = \int_0^3 -x^2 dx \\
&= -\frac{1}{3} x^3 \Big|_0^3 = -9
\end{aligned}$$

$$8.) \vec{F}(x, y) = (x+y)\vec{i} + (-x^2 - y^2)\vec{j}$$

$$\text{a.) Circ} = \oint_C \vec{F} \cdot \vec{T} ds$$

$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R (-2x - 1) dA = \int_0^1 \int_0^x (-2x - 1) dy dx$$



$$\begin{aligned}
 &= \int_0^1 (-2x-1)y \Big|_{y=0}^{y=x} dx \\
 &= \int_0^1 (-2x-1)x dx = \int_0^1 (-2x^2-x) dx \\
 &= \left( -\frac{2}{3}x^3 - \frac{1}{2}x^2 \right) \Big|_0^1 = -\frac{2}{3} - \frac{1}{2} = -\frac{7}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{b.) Flux} &= \oint_C \vec{F} \cdot \vec{n} ds = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA \\
 &= \iint_R (1 + (-2y)) dA = \int_0^1 \int_0^x (1-2y) dy dx \\
 &= \int_0^1 (y - y^2) \Big|_{y=0}^{y=x} dx = \int_0^1 (x - x^2) dx \\
 &= \left( \frac{1}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}
 \end{aligned}$$

$$10.) \vec{F}(x,y) = \left( \arctan\left(\frac{y}{x}\right) \right) \vec{i} + \left( \ln(x^2+y^2) \right) \vec{j}$$

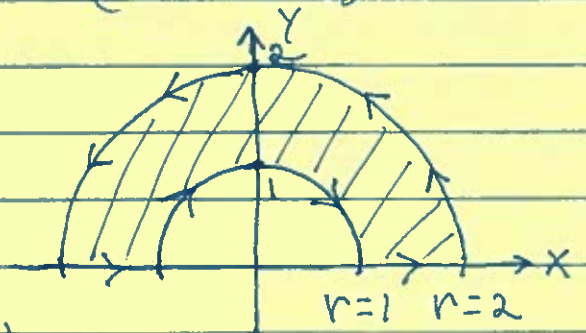
$$\text{a.) Circ} = \oint_C \vec{F} \cdot \vec{T} ds$$

$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R \left( \frac{2x}{x^2+y^2} - \frac{1}{1+\left(\frac{y}{x}\right)^2} \right) dA$$

$$= \iint_R \left( \frac{2x}{x^2+y^2} - \frac{x}{x^2+y^2} \right) dA = \iint_R \frac{x}{x^2+y^2} dA$$

$$= \int_0^\pi \int_1^2 \frac{r \cos \theta}{r^2} \cdot r dr d\theta$$



$$= \int_0^{\pi} r \cos 2\theta \Big|_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi} (2 \cos \theta - 1 \cos 2\theta) d\theta = \int_0^{\pi} \cos \theta d\theta$$

$$= \sin \theta \Big|_0^{\pi} = \sin \pi - \sin 0 = 0.$$

$$b.) \text{ Flux} = \oint_C \vec{F} \cdot \vec{n} ds = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

$$= \iint_R \left( \frac{-y/x^2}{1+(y/x)^2} + \frac{2y}{x^2+y^2} \right) dA$$

$$= \iint_R \left( \frac{-y}{x^2+y^2} + \frac{2y}{x^2+y^2} \right) dA = \iint_R \frac{y}{x^2+y^2} dA$$

$$= \int_0^{\pi} \int_1^2 \frac{r \sin \theta}{r^2} \cdot r dr d\theta = \int_0^{\pi} r \sin \theta \Big|_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi} \sin \theta d\theta = -\cos \theta \Big|_0^{\pi}$$

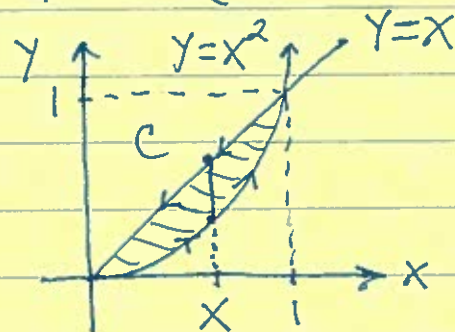
$$= -\cos \pi - (-\cos 0) = -(-1) + 1 = 2$$

$$ii.) \vec{F}(x,y) = (xy)\vec{i} + (y^2)\vec{j}$$

$$a.) \text{ Circ} = \oint_C \vec{F} \cdot \vec{T} ds$$

$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R (0 - x) dA = \int_0^1 \int_{x^2}^x -x dy dx$$





$$\begin{aligned}
 &= \int_0^1 (-xy) \Big|_{y=x^2}^{y=x} dx = \int_0^1 [-x^2 - (-x^3)] dx \\
 &= \int_0^1 (x^3 - x^2) dx = \left( \frac{1}{4}x^4 - \frac{1}{3}x^3 \right) \Big|_0^1 \\
 &= \frac{1}{4} - \frac{1}{3} = \frac{3}{12} - \frac{4}{12} = -\frac{1}{12}
 \end{aligned}$$

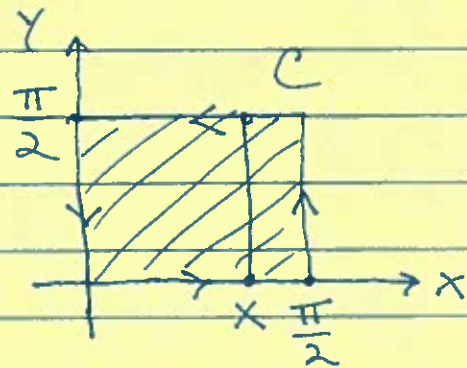
$$b.) \text{ Flux} = \oint_C \vec{F} \cdot \vec{n} ds$$

$$= \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA = \iint_R (y + 2y) dA$$

$$= \int_0^1 \int_{x^2}^x 3y dy dx = \int_0^1 \left( \frac{3}{2}y^2 \Big|_{y=x^2}^{y=x} \right) dx$$

$$\begin{aligned}
 &= \int_0^1 \left( \frac{3}{2}x^2 - \frac{3}{2}x^4 \right) dx = \left( \frac{1}{2}x^3 - \frac{3}{10}x^5 \right) \Big|_0^1 \\
 &= \frac{1}{2} - \frac{3}{10} = \frac{5}{10} - \frac{3}{10} = \frac{2}{10} = \frac{1}{5}
 \end{aligned}$$

$$12.) \vec{F}(x,y) = (-\sin y) \vec{i} + (x \cos y) \vec{j}$$



$$a.) \text{ Circ} = \oint_C \vec{F} \cdot \vec{T} ds$$

$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_R (\cos y - (-\cos y)) dA$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} 2 \cos y dy dx = \int_0^{\pi/2} \left( 2 \sin y \Big|_{y=0}^{y=\pi/2} \right) dx$$

$$= \int_0^{\pi/2} (2 \sin \frac{\pi}{2} - 2 \sin 0) dx = 2x \Big|_0^{\pi/2} = \pi$$

$$\begin{aligned}
 \text{b.) Flux} &= \oint_C \vec{F} \cdot \vec{n} \, ds = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA \\
 &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (0 + (-x \sin y)) \, dy \, dx \\
 &= \int_0^{\frac{\pi}{2}} \left( x \cos y \Big|_{y=0}^{y=\frac{\pi}{2}} \right) dx \\
 &= \int_0^{\frac{\pi}{2}} \left( x \cos \frac{\pi}{2} - x \cos 0 \right) dx \\
 &= \int_0^{\frac{\pi}{2}} -x \, dx = -\frac{1}{2} x^2 \Big|_0^{\frac{\pi}{2}} = -\frac{1}{8} \pi^2
 \end{aligned}$$

$$13.) \vec{F}(x,y) = \left( 3xy - \frac{x}{1+y^2} \right) \vec{i} + \left( e^x + \arctan y \right) \vec{j},$$

$$\text{Flux} = \oint_C \vec{F} \cdot \vec{n} \, ds$$

$$= \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

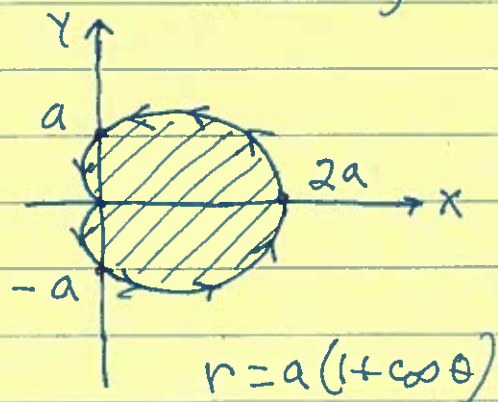
$$= \iint_R \left( \left( 3y - \frac{1}{1+y^2} \right) + \frac{1}{1+y^2} \right) dA$$

$$= \int_0^{2\pi} \int_0^{a(1+\cos\theta)} 3 \cdot r \sin\theta \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{a(1+\cos\theta)} 3r^2 \cdot \sin\theta \, dr \, d\theta$$

$$= \int_0^{2\pi} \left( r^3 \sin\theta \Big|_{r=0}^{r=a(1+\cos\theta)} \right) d\theta$$

$$= \int_0^{2\pi} a^3 (1+\cos\theta)^3 \sin\theta \, d\theta$$



$$= -\frac{1}{4} a^3 (1 + \cos 2\theta) \Big|_0^{2\pi}$$

$$= -\frac{1}{4} a^3 (1 + \cos 2\pi) - \left(-\frac{1}{4} a^3 (1 + \cos 2 \cdot 0)\right) = 0$$

15.)  $\text{Work} = \oint_C \vec{F} \cdot \vec{T} ds$

$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

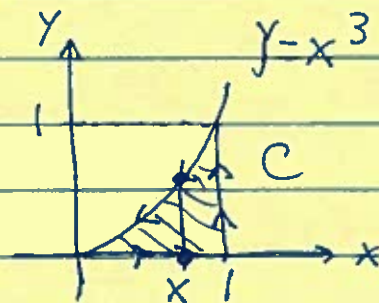
$$= \iint_R (8xy^2 - 6xy^2) dA$$

$$\vec{F}(x,y) = (2xy^3)\vec{i} + (4x^2y^2)\vec{j}$$

$$= \int_0^1 \int_0^{x^3} 2xy^2 dy dx = \int_0^1 \left( 2x \cdot \frac{1}{3} y^3 \Big|_{y=0}^{y=x^3} \right) dx$$

$$= \int_0^1 \frac{2}{3} x \cdot x^9 dx = \int_0^1 \frac{2}{3} x^{10} dx = \frac{2}{3} \cdot \frac{1}{11} x^{11} \Big|_0^1$$

$$= \frac{2}{33}$$



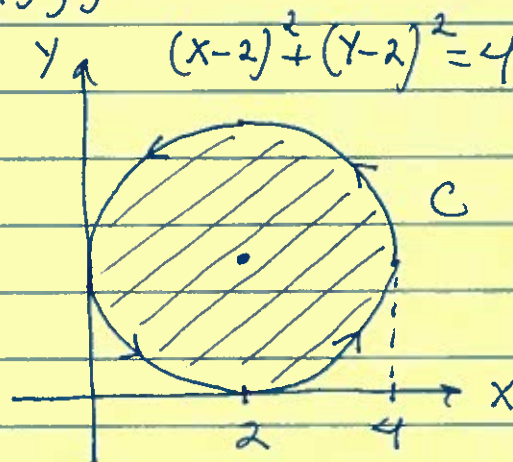
16.)  $\vec{F}(x,y) = (4x - 2y)\vec{i} + (2x - 4y)\vec{j}$

$$\text{Work} = \oint_C \vec{F} \cdot \vec{T} ds$$

$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R (2 - (-2)) dA$$

$$= 4 \iint_R 1 dA = 4 (\text{area } R) = 4 \cdot \pi (2)^2 = 16\pi$$



$$17.) \oint_C y^2 dx + x^2 dy$$

$$(\vec{F}(x,y) = (y^2)\vec{i} + (x^2)\vec{j})$$

$$= \oint_C \vec{F} \cdot \vec{T} ds = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

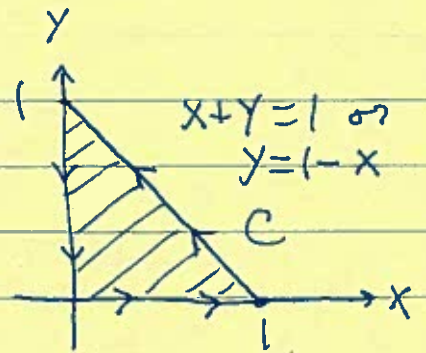
$$= \iint_R (2x - 2y) dA = \int_0^1 \int_0^{1-x} (2x - 2y) dy dx$$

$$= \int_0^1 (2xy - y^2) \Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 [2x(1-x) - (1-x)^2] dx = \int_0^1 [2x - 2x^2 - (x^2 - 2x + 1)] dx$$

$$= \int_0^1 (-3x^2 + 4x - 1) dx = \left( -x^3 + 2x^2 - x \right) \Big|_0^1$$

$$= -1 + 2 - 1 = 0$$



$$18.) \oint_C 3y dx + 2x dy$$

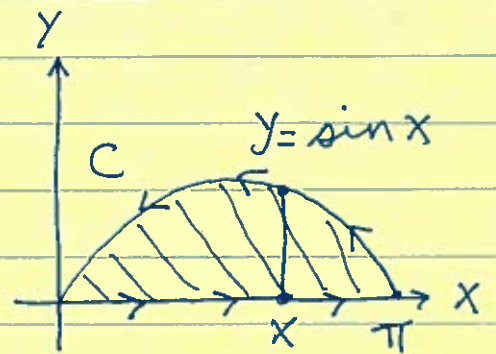
$$(\vec{F}(x,y) = (3y)\vec{i} + (2x)\vec{j})$$

$$= \oint_C \vec{F} \cdot \vec{T} ds = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R (2 - 3) dA = \int_0^\pi \int_0^{\sin x} -1 dy dx$$

$$= \int_0^\pi \left( -y \Big|_{y=0}^{y=\sin x} \right) dx = \int_0^\pi -\sin x dx$$

$$= \cos x \Big|_0^\pi = \cos \pi - \cos 0 = -1 - 1 = -2$$



$$19.) \oint_C (6y+x) dx + (y+2x) dy$$

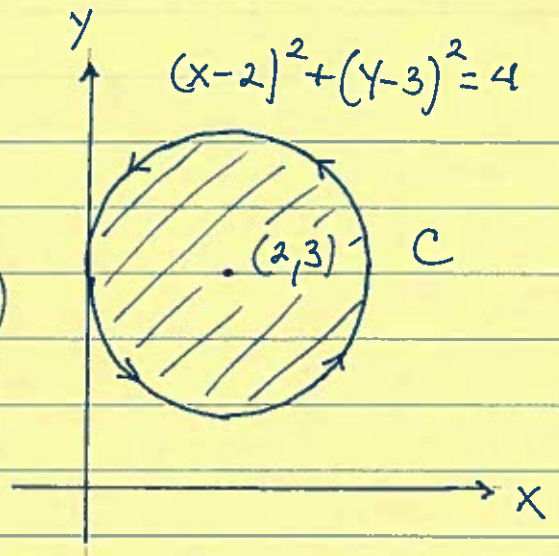
$$(\vec{F}(x,y) = (6y+x)\vec{i} + (y+2x)\vec{j})$$

$$= \oint_C \vec{F} \cdot \vec{T} ds$$

$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R (2-6) dA = -4 \iint_R 1 dA$$

$$= -4 (\text{area } R) = -4 \cdot \pi(2)^2 = -16\pi$$



$$20.) \oint_C (2x+y^2) dx + (2xy+3y) dy$$

$$(\vec{F}(x,y) = (2x+y^2)\vec{i} + (2xy+3y)\vec{j})$$

$$= \oint_C \vec{F} \cdot \vec{T} ds \quad (\text{and } M_y = 2y = N_x,$$

so  $\vec{F}$  is conservative)

$$= 0 \quad (\text{since } C \text{ is a closed curve})$$

$$22.) \vec{r}(t) = (a \cos t)\vec{i} + (b \sin t)\vec{j}, \quad 0 \leq t \leq 2\pi$$

$$C: \begin{cases} x = a \cos t \\ y = b \sin t \end{cases} \quad \text{for } 0 \leq t \leq 2\pi$$

$$\frac{dx}{dt} = -a \sin t, \quad \frac{dy}{dt} = b \cos t$$

$$\text{Area} = \frac{1}{2} \oint_C x dy - y dx$$

$$= \frac{1}{2} \int_0^{2\pi} \left[ (a \cos t) \frac{dy}{dt} - (b \sin t) \frac{dx}{dt} \right] dt$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^{2\pi} [(a \cos t)(b \cos t) - (b \sin t)(-a \sin t)] dt \\
&= \frac{1}{2} \int_0^{2\pi} ab (\underbrace{\cos^2 t + \sin^2 t}_1) dt \\
&= \frac{1}{2} ab t \Big|_0^{2\pi} = ab\pi
\end{aligned}$$

24.)  $C: \begin{cases} x = t^2 \\ y = \frac{1}{3}t^3 - t \end{cases}$  for  $-\sqrt{3} \leq t \leq \sqrt{3}$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = t^2 - 1, \quad \text{then}$$

$$\text{Area} = \frac{1}{2} \oint_C x dy - y dx$$

$$= \frac{1}{2} \oint_C [t^2 \cdot (t^2 - 1) - (\frac{1}{3}t^3 - t)(2t)] dt$$

$$= \frac{1}{2} \int_{-\sqrt{3}}^{\sqrt{3}} [t^4 - t^2 - (\frac{2}{3}t^4 - 2t^2)] dt$$

$$= \frac{1}{2} \int_{-\sqrt{3}}^{\sqrt{3}} [\frac{1}{3}t^4 + t^2] dt$$

$$= \frac{1}{2} \left( \frac{1}{15} \cdot 9\sqrt{3} + \frac{1}{3} \cdot 3\sqrt{3} \right) - \frac{1}{2} \left( \frac{1}{15} (-9\sqrt{3}) + \frac{1}{3} (-3\sqrt{3}) \right)$$

$$= \frac{1}{30} 9\sqrt{3} + \frac{1}{2} \sqrt{3} + \frac{9}{30} \sqrt{3} + \frac{1}{2} \sqrt{3}$$

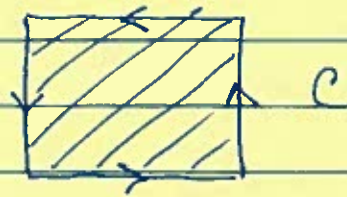
$$= \frac{18}{30} \sqrt{3} + \sqrt{3} = \frac{3}{5} \sqrt{3} + \frac{5}{5} \sqrt{3} = \frac{8}{5} \sqrt{3}$$

26.)  $\oint_C (xy^2) dx + (x^2y + 2x) dy$

$$(\vec{F}(x,y) = (xy^2)\vec{i} + (x^2y + 2x)\vec{j})$$

Square of area A

$$= \oint_C \vec{F} \cdot \vec{T} ds$$



$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R ((2xy+2) - (2xy)) dA$$

$$= 2 \cdot \iint_R 1 dA = 2 \cdot (\text{area of square})$$

$$= 2A \quad (\text{independent of location of square})$$

29.) Since

$$\oint_C \vec{F} \cdot \vec{u} ds = \oint_C M dy - N dx = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA,$$

a.) let  $\vec{F}(x,y) = (x)\vec{i} + (0)\vec{j}$  then

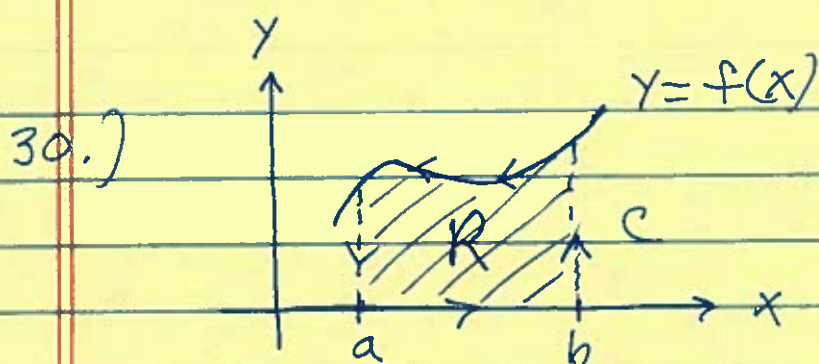
$$\oint_C x dy - (0) dx = \iint_R 1 dA \rightarrow$$

$$\text{Area } R = \oint_C x dy.$$

b.) let  $\vec{F}(x,y) = (0)\vec{i} + (y)\vec{j}$  then

$$\oint_C (0) dy - y dx = \iint_R 1 dA \rightarrow$$

$$\text{Area } R = -\oint_C y dx.$$



Let  $C$  be loop around region, then by problem 29

$$\int_a^b f(x) dx = \text{Area } R = - \oint_C y dx$$

33.) Assume that  $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ .

Show that  $\oint_C \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy = 0$  :

$$\oint_C \frac{\partial f}{\partial y} dx + \left( -\frac{\partial f}{\partial x} \right) dy \rightarrow$$

$$\vec{F} = \underbrace{\left( \frac{\partial f}{\partial y} \right)}_M \vec{i} + \underbrace{\left( -\frac{\partial f}{\partial x} \right)}_N \vec{j} \quad \text{so that}$$

$$M_y = \frac{\partial^2 f}{\partial y^2}, \quad N_x = -\frac{\partial^2 f}{\partial x^2}; \quad \text{then}$$

$$\oint_C \frac{\partial f}{\partial y} dx + \left( -\frac{\partial f}{\partial x} \right) dy = \oint_C \vec{F} \cdot \vec{T} ds$$

$$= \iint_R (N_x - M_y) dA = \iint_R \left( -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} \right) dA$$

$$= - \iint_R \underbrace{\left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right)}_0 dA = 0$$