Section 15.6

21.) $\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$= \int_0^{\pi}\int_0^{\pi} \left(\frac{1}{3}\rho^3 \sin\phi \Big|_{\rho=0}^{\rho=2\sin\phi}\right) d\phi \, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} \frac{1}{3}(8\sin^3\phi)\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} \frac{8}{3}\sin^4\phi \, d\phi \, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} \frac{8}{3}(\sin^2\phi)^2 \, d\phi \, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} \frac{8}{3}\left(\frac{1}{2}(1-\cos 2\phi)\right)^2 d\phi \, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} \frac{8}{3}\cdot\frac{1}{4}(1-2\cos 2\phi+\cos^2 2\phi) \, d\phi \, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} \frac{2}{3}\left(1-2\cos 2\phi+\frac{1}{2}(1+\cos 4\phi)\right) d\phi \, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} \frac{2}{3}\left(\frac{3}{2}-2\cos 2\phi+\frac{1}{2}\cos 4\phi\right) d\phi \, d\theta$$

$$= \int_0^{\pi} \frac{2}{3}\left(\frac{3}{2}\phi-2\cdot\frac{1}{2}\sin 2\phi+\frac{1}{2}\cdot\frac{1}{4}\sin 4\phi\right)\Big|_{\phi=0}^{\phi=\pi} d\theta$$

$$= \int_0^{\pi} \frac{2}{3}\left(\frac{3}{2}\pi-\overset{0}{\cancel{\sin 2\pi}}+\frac{1}{8}\overset{0}{\cancel{\sin 4\pi}}\right)-\frac{1}{3}(0) \quad d\theta$$

$$= \int_0^{\pi} \pi \, d\theta = \pi\,\theta\Big|_0^{\pi} = \pi(\pi-0) = \pi^2$$

24.) $\int_0^{\frac{3}{2}\pi}\int_0^{\pi}\int_0^{1} 5\rho^3\sin^3\phi \, d\rho \, d\phi \, d\theta$

$$= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \left(\frac{5}{4} \rho^4 \sin^3\phi \Big|_{\rho=0}^{\rho=1} \right) d\phi \, d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \frac{5}{4} \sin^3\phi \, d\phi \, d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \frac{5}{4} \sin\phi \cdot \sin^2\phi \, d\phi \, d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \frac{5}{4} \sin\phi \left(1 - \cos^2\phi\right) d\phi \, d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \frac{5}{4} \left(\sin\phi - \sin\phi \cos^2\phi \right) d\phi \, d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \frac{5}{4} \left(-\cos\phi + \frac{1}{3}\cos^3\phi \right) \Big|_{\phi=0}^{\phi=\pi} d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \left[\frac{5}{4} \left(-\cos\pi + \frac{1}{3}\cos^3\pi \right) - \frac{5}{4} \left(-\cos 0 + \frac{1}{3}\cos^3 0 \right) \right] d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \left[\frac{5}{4} \left(-(-1) + \frac{1}{3}(-1) \right) - \frac{5}{4} \left(-1 + \frac{1}{3}(1) \right) \right] d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \left(\frac{5}{\cancel{4}_2} \cdot \frac{\cancel{2}}{3} - \frac{5}{\cancel{4}_2} \cdot \frac{-\cancel{2}}{3} \right) d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \frac{10}{6} \, d\theta = \int_0^{3/2\pi} \frac{5}{3} \, d\theta$$

$$= \frac{5}{3} \theta \Big|_0^{\frac{3}{2}\pi} = \frac{5}{\cancel{3}} \cdot \frac{\cancel{3}}{2} \pi = \frac{5}{2}\pi$$

26.) $$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec\phi} \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left(\frac{1}{4} \rho^4 \cdot \cos\phi \sin\phi \Big|_{\rho=0}^{\rho=\sec\phi} \right) d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{4} \sec^4\phi \cos\phi \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{4} \sec^2\phi \cdot \frac{1}{\cancel{\cos\phi}} \cancel{\cos\phi} \cdot \frac{1}{\cos\phi} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{4} \sec^2\phi \tan\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4} \cdot \frac{1}{2} \tan^2\phi \Big|_{\phi=0}^{\phi=\frac{\pi}{4}} \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{8} \tan^2 \frac{\pi}{4} - \frac{1}{8} \overset{0}{\cancel{\tan^2 0}} \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{8} (1)^2 \, d\theta = \frac{1}{8} \theta \Big|_0^{2\pi} = \frac{\pi}{4}$$

27.) $$\int_0^2 \int_{-\pi}^0 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho$$

$$= \int_0^2 \int_{-\pi}^0 \left(\rho^3 \cdot \frac{-1}{2} \cos 2\phi \Big|_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{2}} \right) d\theta \, d\rho$$

$$= \int_0^2 \int_{-\pi}^0 \left(\frac{-1}{2} \rho^3 \overset{-1}{\cancel{\cos \pi}} - \frac{-1}{2} \rho^3 \overset{0}{\cancel{\cos \frac{\pi}{2}}} \right) d\theta \, d\rho$$

$$= \int_0^2 \int_{-\pi}^0 \frac{1}{2} \rho^3 \, d\theta \, d\rho = \int_0^2 \left(\frac{1}{2} \rho^3 \cdot \theta \Big|_{\theta=-\pi}^{\theta=0} \right) d\rho$$

$$= \int_0^2 \left(\frac{1}{2} \rho^3 \cdot (0) - \frac{1}{2} \rho^3 (-\pi) \right) d\rho$$

$$= \int_0^2 \frac{\pi}{2} \rho^3 \, d\rho = \frac{\pi}{2} \cdot \frac{1}{4} \rho^4 \Big|_0^2$$

$$= \frac{\pi}{2} \cdot \frac{1}{4} \cdot 16 = 2\pi$$

30.) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\rho^5 \cdot \sin^3\phi \Big|_{\rho=\csc\phi}^{\rho=2} \right) d\theta \, d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(32 \sin^3\phi - \csc^5\phi \cdot \sin^3\phi \right) d\theta \, d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(32 \sin^3\phi - \csc^2\phi \cdot \frac{1}{\cancel{\sin^3\phi}} \cdot \cancel{\sin^3\phi} \right) d\theta \, d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(32 \sin^3\phi - \csc^2\phi \right) \cdot \theta \Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \, d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(32 \sin^3\phi - \csc^2\phi \right) \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \pi \left(32 \cdot \sin\phi \cdot \sin^2\phi - \csc^2\phi \right) d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \pi \left(32 \cdot \sin\phi \left(1 - \cos^2\phi \right) - \csc^2\phi \right) d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \pi \left(32 \sin\phi - 32 \sin\phi \cos^2\phi - \csc^2\phi \right) d\phi$$

$$= \pi \left(-32 \cos\phi + 32 \cdot \frac{1}{3} \cos^3\phi + \cot\phi \right) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \pi\left(-32\cancelto{0}{\cos\frac{\pi}{2}} + \frac{32}{3}\cancelto{0}{\cos^3\frac{\pi}{2}} + \cancelto{0}{\cot\frac{\pi}{2}}\right)$$

$$-\pi\left(-32\cos\frac{\pi}{6} + \frac{32}{3}\cos^3\frac{\pi}{6} + \cot\frac{\pi}{6}\right)$$

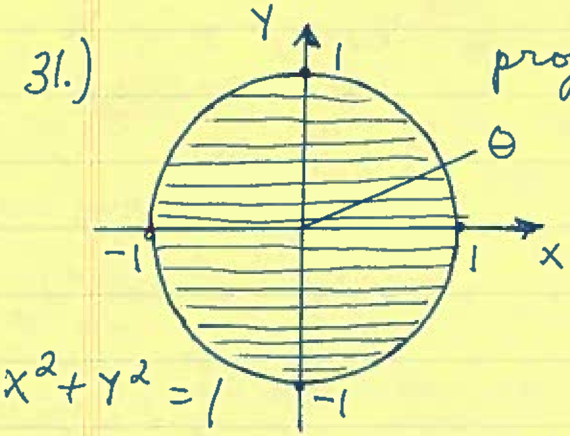
$$= -\pi\left(-32\cdot\frac{\sqrt{3}}{2} + \frac{32}{3}\cdot\left(\frac{\sqrt{3}}{2}\right)^3 + \frac{\sqrt{3}/2}{1/2}\right)$$

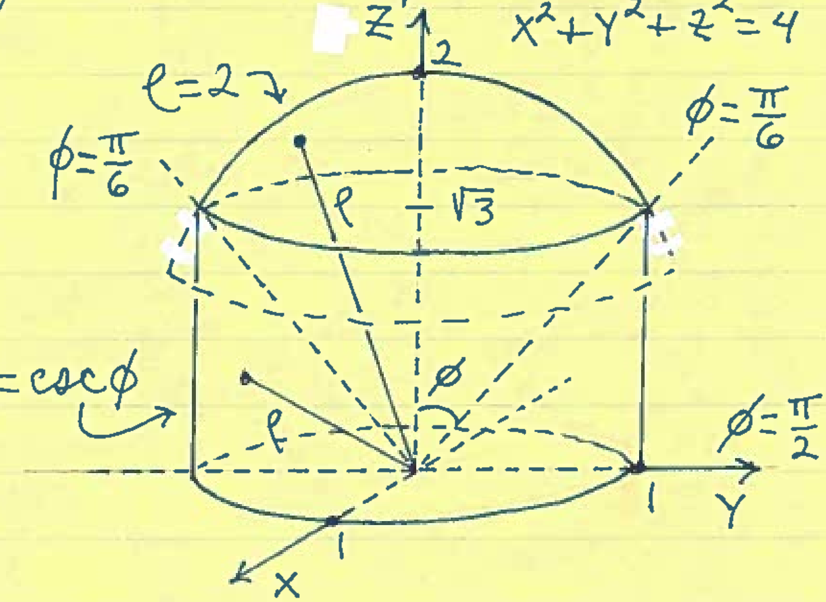
$$= -\pi\left(-16\sqrt{3} + \frac{\overset{4}{\cancel{32}}}{\cancel{3}}\cdot\frac{\cancel{3}\sqrt{3}}{\cancel{8}} + \sqrt{3}\right)$$

$$= -\pi\left(-11\sqrt{3}\right) = 11\sqrt{3}\cdot\pi$$

31.)

projection on XY-plane





$x^2+y^2+z^2=4$

and $x^2+y^2=1 \rightarrow$

$1+z^2=4 \rightarrow z^2=3$

$\rightarrow z=\sqrt{3}$;

$x^2+y^2=1 \rightarrow r=1 \rightarrow \rho\sin\phi=1 \rightarrow \rho=\csc\phi$

$$\text{a.) Vol} = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{2} 1\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$+ \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{\csc\phi} 1\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

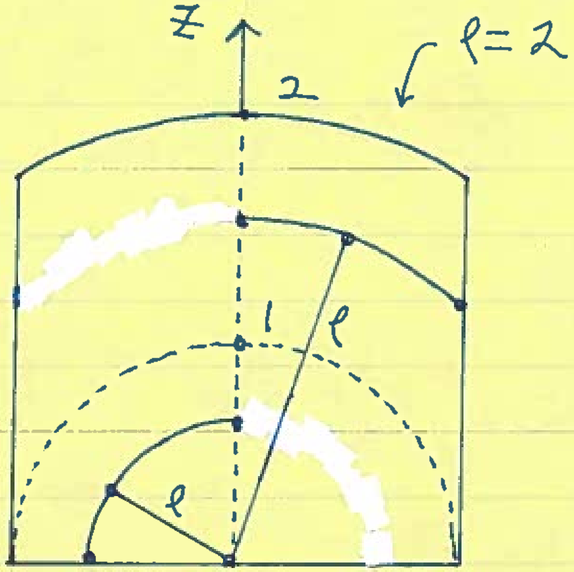
b.) SIDE VIEW OF SOLID

$r = 1 \rightarrow$

$\rho \sin\phi = 1 \rightarrow$

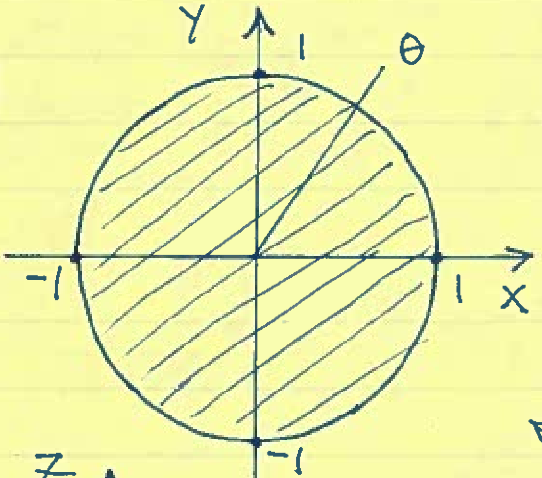
$\sin\phi = \frac{1}{\rho} \rightarrow$

$\phi = \arcsin\left(\frac{1}{\rho}\right)$



$$\text{Vol} = \int_0^{2\pi}\int_0^{1}\int_0^{\frac{\pi}{2}} 1\cdot\rho^2\sin\phi \, d\phi \, d\rho \, d\theta$$
$$+ \int_0^{2\pi}\int_1^{2}\int_0^{\arcsin\left(\frac{1}{\rho}\right)} 1\cdot\rho^2\sin\phi \, d\phi \, d\rho \, d\theta$$

32.)

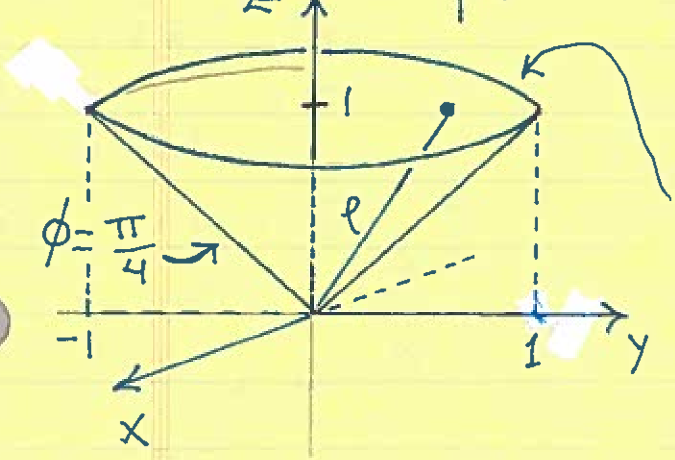


$Z = \sqrt{X^2+Y^2}$ and $Z = 1$

$\rightarrow 1 = \sqrt{X^2+Y^2} \rightarrow$

$X^2 + Y^2 = 1$;

← projection on XY-plane ;



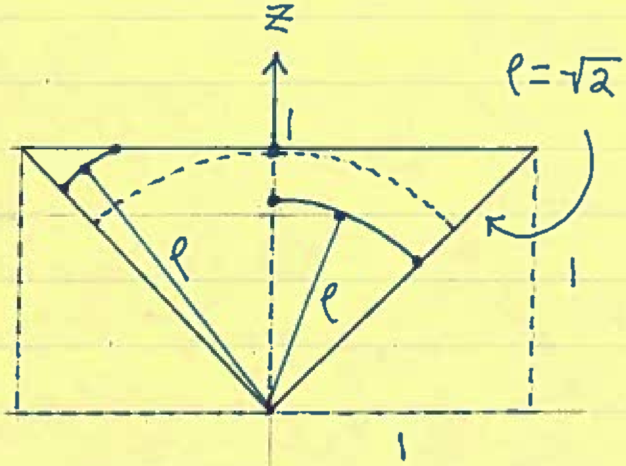
$Z = 1 \rightarrow \rho\cos\phi = 1 \rightarrow$

$\rho = 1/\cos\phi \rightarrow$

$\rho = \sec\phi$

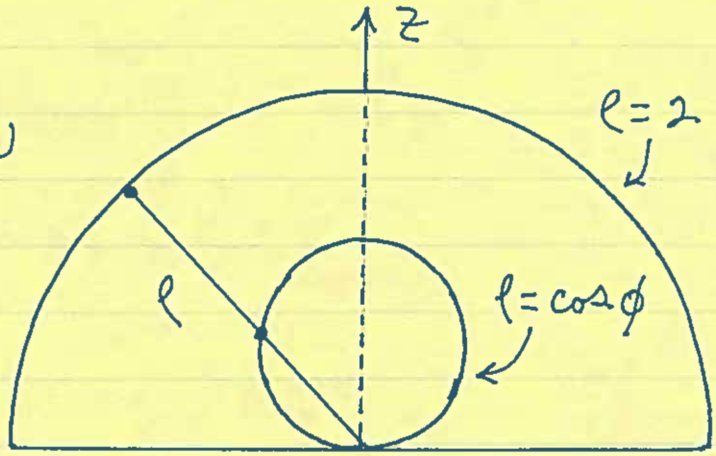
a.) $\text{Vol} = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec\phi} 1 \cdot \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta$

b.) SIDE VIEW
OF SOLID



$$\text{Vol} = \int_0^{2\pi} \int_0^{1} \int_0^{\frac{\pi}{4}} 1 \cdot \rho^2 \sin\phi \; d\phi \, d\rho \, d\theta$$
$$+ \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{\sec\phi}^{\frac{\pi}{4}} 1 \cdot \rho^2 \sin\phi \; d\phi \, d\rho \, d\theta$$

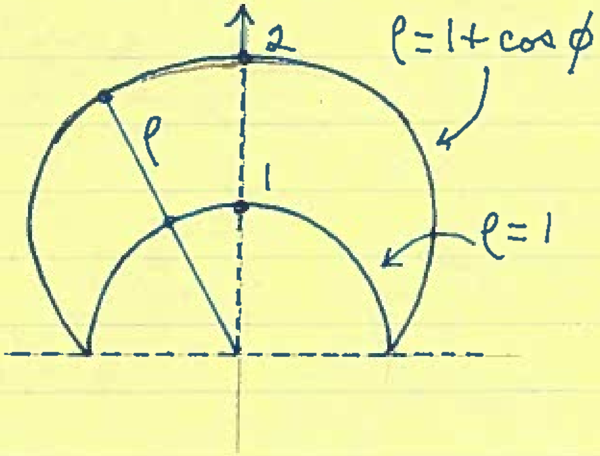
33.) SIDE VIEW
OF SOLID



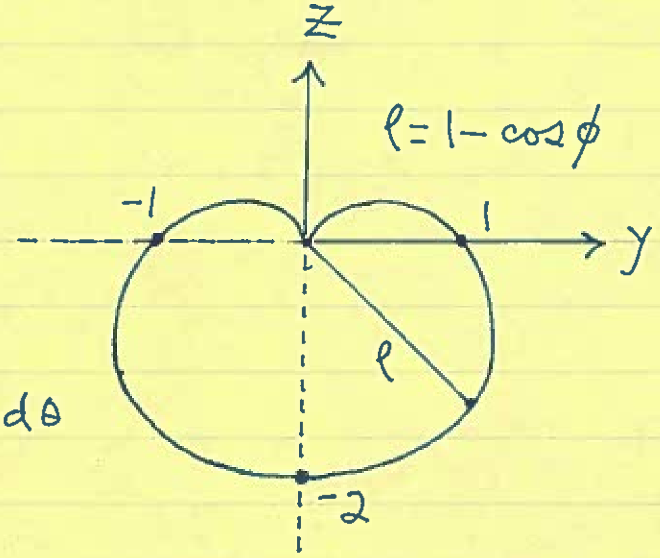
$$\text{Vol} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_{\cos\phi}^{2} 1 \cdot \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta$$

34.)

$$\text{Vol} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^{1+\cos\phi} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



35.) SIDE VIEW

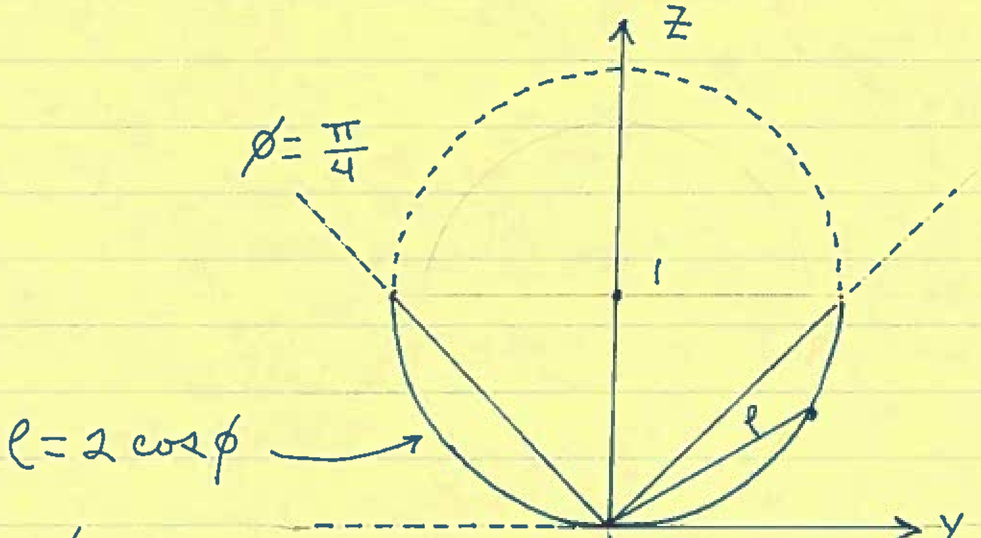


$$\text{Vol} = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} 1\cdot\rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

37.) $z = \sqrt{x^2+y^2}$ (cone)

$\rightarrow \phi = \frac{\pi}{4}$

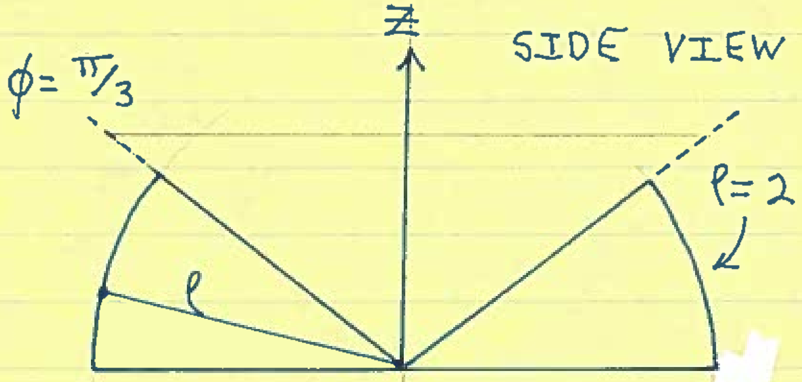
SIDE VIEW



$$\text{Vol} = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\cos\phi} 1\cdot\rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

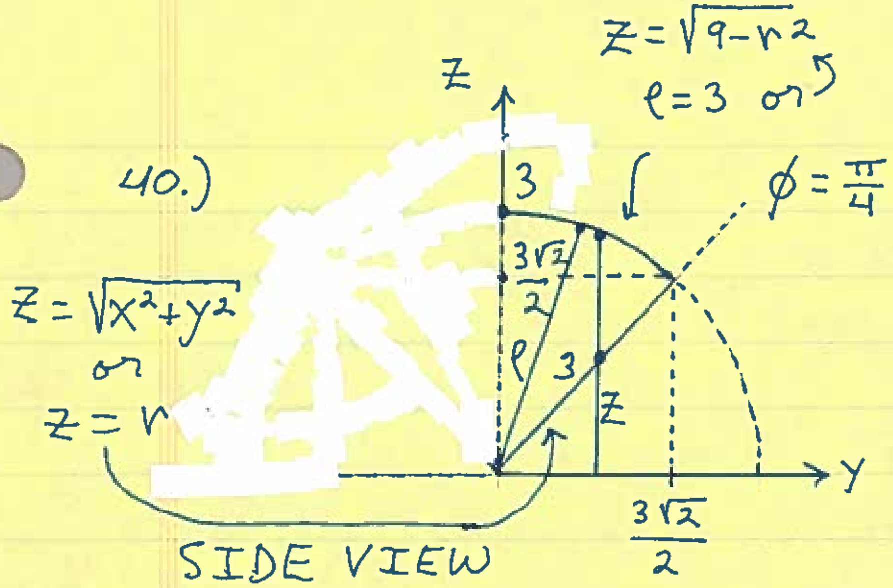
38.)

SIDE VIEW

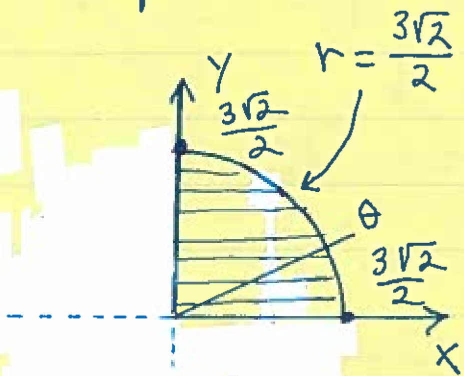


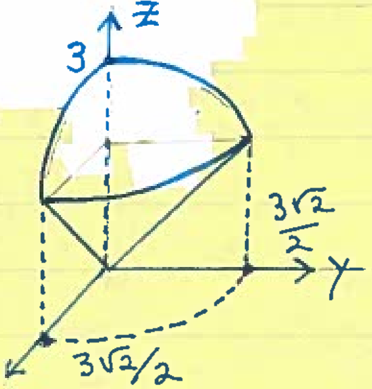
$$\text{Vol} = \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{2} 1\cdot\rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

40.)



proj. on xy-plane

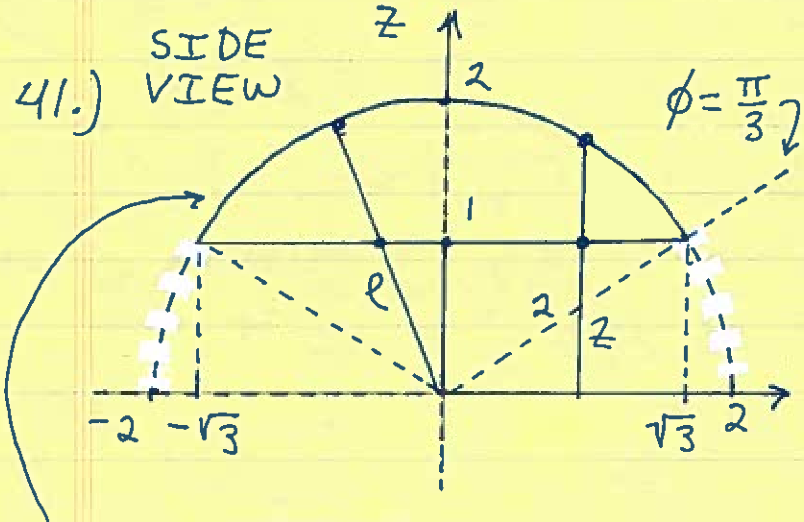




a.) $Vol = \int_0^{\frac{\pi}{2}} \int_0^{\frac{3\sqrt{2}}{2}} \int_r^{\sqrt{9-r^2}} 1 \cdot r \, dz \, dr \, d\theta$

b.) $Vol = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{3} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

41.)



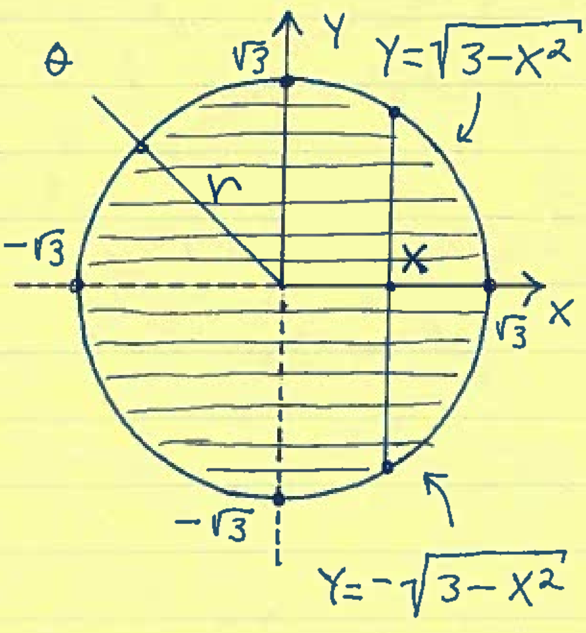
$z = -\sqrt{4-x^2-y^2}$ or

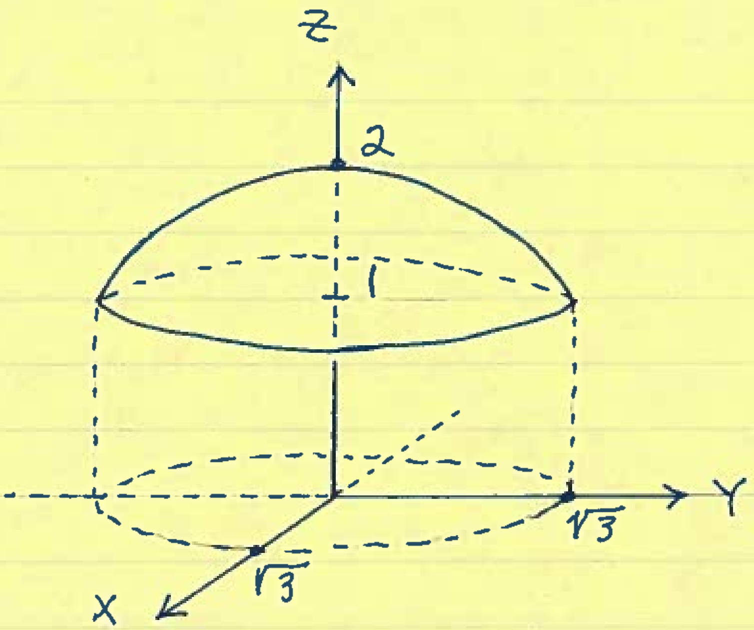
$z = \sqrt{4-r^2}$ or

$\rho = 2$;

$z = 1$ or $\rho\cos\phi = 1$ or $\rho = \sec\phi$

proj. on xy-plane



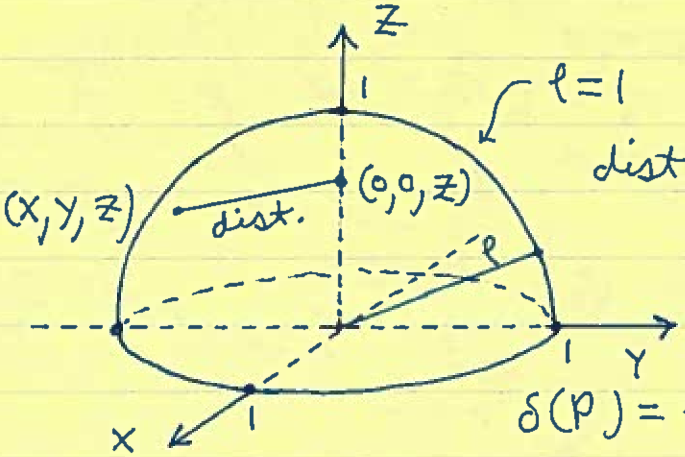


a.) $$\text{Vol} = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec\phi}^{\rho} 1\cdot\rho^2\sin\phi\; d\rho\; d\phi\; d\theta$$

b.) $$\text{Vol} = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} 1\cdot r\; dz\; dr\; d\theta$$

c.) $$\text{Vol} = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} 1\; dz\; dY\; dx$$

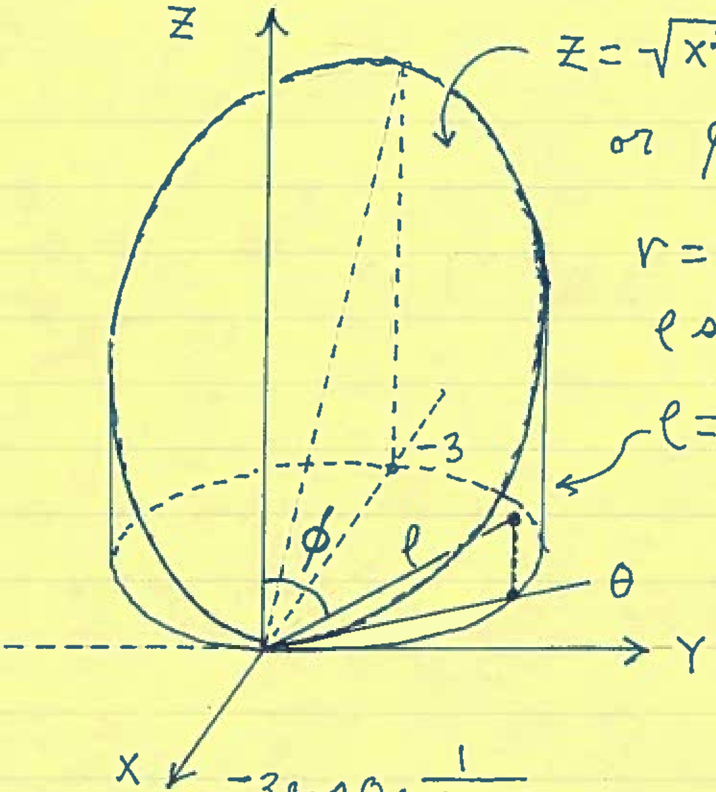
42.) b.)



$$\text{dist} = \sqrt{(x-0)^2+(Y-0)^2+(z-z)^2}$$
$$= \sqrt{x^2+y^2} = \sqrt{r^2}$$
$$= r = \rho\sin\phi,$$
$$\delta(P) = z = \rho\cos\phi;$$

$$I_z = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 (\rho\sin\phi)^2\cdot(\rho\cos\phi)\cdot\rho^2\sin\phi\cdot d\rho\, d\phi\, d\theta$$

46.)



$z = \sqrt{x^2+y^2}$ (cone)

or $\phi = \frac{\pi}{4}$

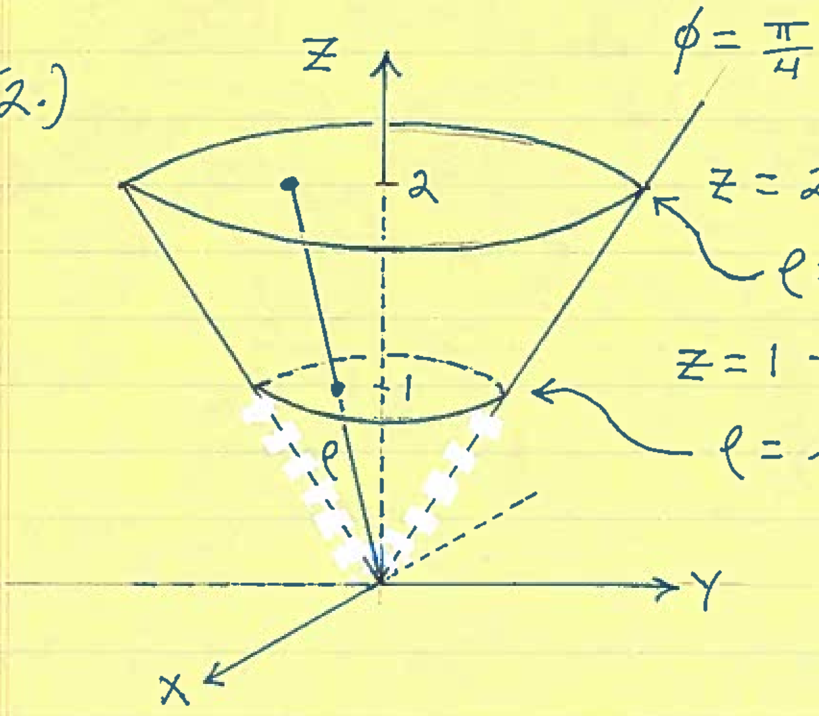
$r = -3\cos\theta \rightarrow$

$\rho \sin\phi = -3\cos\theta \rightarrow$

$\rho = -3\cos\theta \cdot \frac{1}{\sin\phi}$

$$\text{Vol} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{0}^{-3\cos\theta \cdot \frac{1}{\sin\phi}} 1 \cdot \rho^2 \sin\phi \cdot d\rho \, d\phi \, d\theta$$

52.)



$z = 2 \rightarrow \rho\cos\phi = 2 \rightarrow$

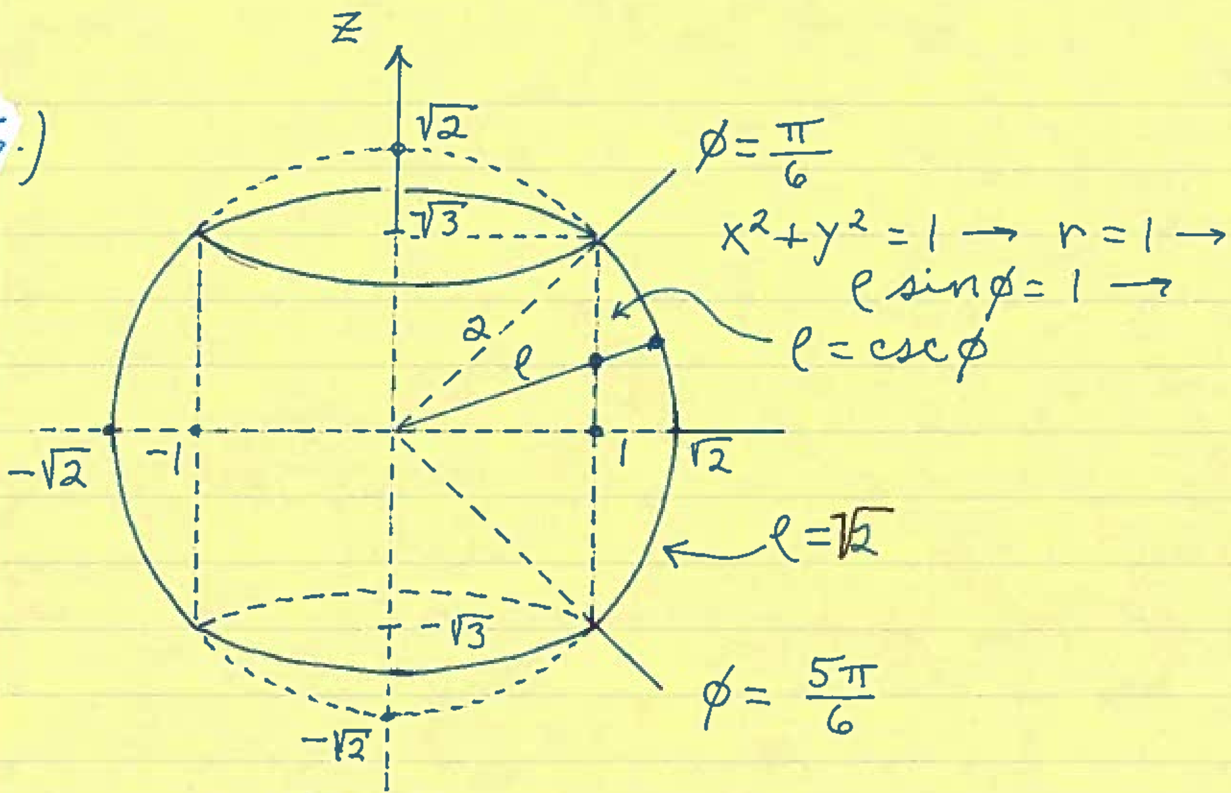
$\rho = 2\sec\phi$

$z = 1 \rightarrow \rho\cos\phi = 1 \rightarrow$

$\rho = \sec\phi$

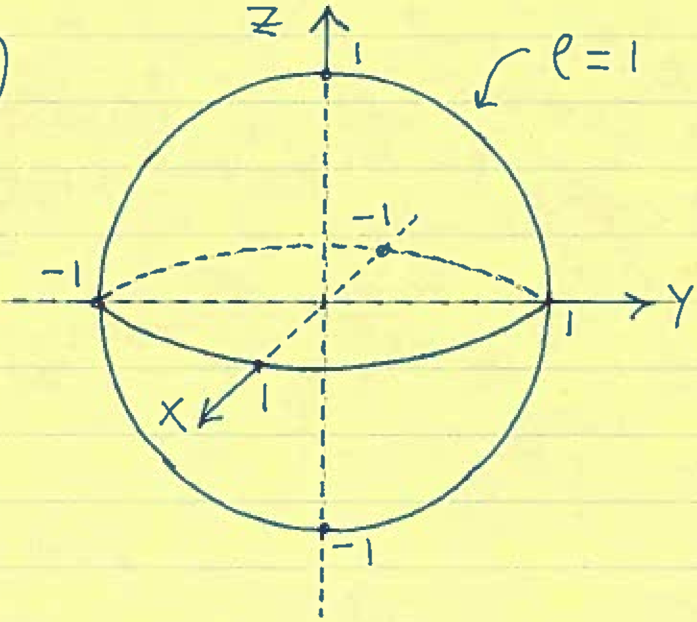
$$\text{Vol} = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_{\sec\phi}^{2\sec\phi} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

56.)



$$\text{Vol} = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{\csc\phi}^{\sqrt{2}} 1\cdot \rho^2\sin\phi \cdot d\rho\, d\phi\, d\theta$$

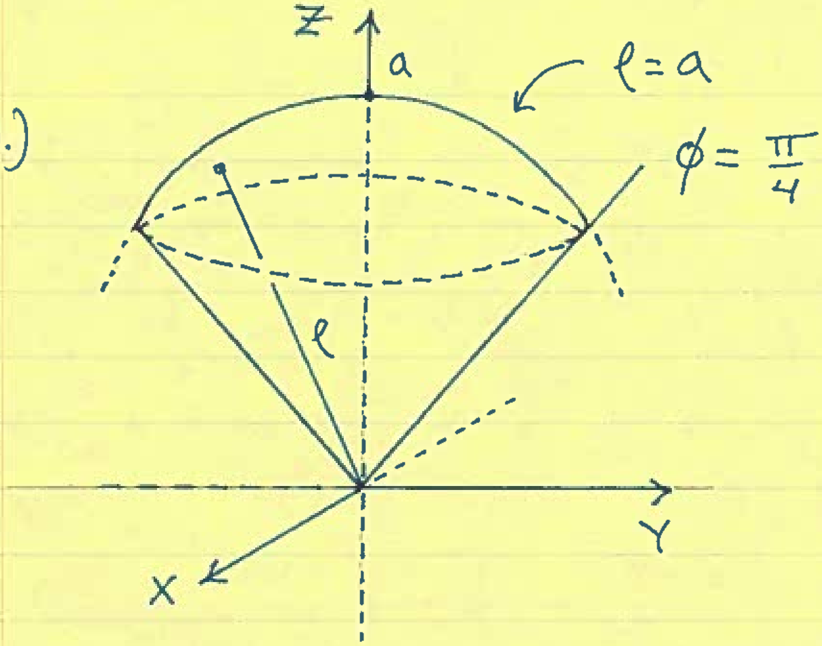
65)



$$\text{AVE} = \frac{1}{\text{vol } R} \iiint_R f(P)\, dV$$

$$= \frac{1}{\int_0^{2\pi}\int_0^{\pi}\int_0^{1} 1\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta} \cdot \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

70.)



$$\bar{X} = \iiint_R x \, dV \Big/ \iiint_R 1 \, dV$$

$$= \frac{\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{a} (\rho \cdot \sin\phi \cos\theta)\cdot \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta}{\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{a} 1 \cdot \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta}$$