Section 13.2
Thomas Calculus
11th Edition

Integrals of Vector Functions

Definition: The Definite Integral of vector function

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \text{ is}$$

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}.$$

Example: Integrate each vector function.

1.) $\int_0^1 \left[(e^{2t})\vec{i} - (\cos t)\vec{j} + 5\vec{k} \right] dt$

$= \left(\frac{1}{2}e^{2t}\Big|_0^1\right)\vec{i} - \left(\sin t\Big|_0^1\right)\vec{j} + \left(5t\Big|_0^1\right)\vec{k}$

$= \left(\frac{1}{2}e^2 - \frac{1}{2}\right)\vec{i} - (\sin 1)\vec{j} + 5\vec{k}$

2.) $\displaystyle\int_1^e \left[(\ln t)\vec{i} + \left(\frac{1}{t}\right)\vec{j} - (3t^2)\vec{k}\right] dt$

Integration by Parts

$= (t\ln t - t)\Big|_1^e \vec{i} + (\ln t)\Big|_1^e \vec{j} - \left(t^3\Big|_1^e\right)\vec{k}$

$= \left[(e\overset{1}{\cancel{\ln e}} - e) - (1\overset{0}{\cancel{\ln 1}} - 1)\right]\vec{i}$

$(\underset{1}{\cancel{\ln e}} - \underset{0}{\cancel{\ln 1}})\vec{j} - (e^3 - 1^3)\vec{k}$

$= (1-e)\vec{i} + \vec{j} - (e^3 - 1)\vec{k}$

Example: Solve the following Initial Value Problems (Differential Vector Equations).

1.) $\dfrac{d}{dt}\vec{r}(t) = \vec{r}\,'(t) = (1)\vec{i} + (t)\vec{j} - (t^2)\vec{k}$

and $\vec{r}(0) = 2\vec{i} - 4\vec{j} + 9\vec{k}$;

$\xrightarrow{\int} \vec{r}(t) = (t + c_1)\vec{i} + \left(\frac{1}{2}t^2 + c_2\right)\vec{j} - \left(\frac{1}{3}t^3 + c_3\right)\vec{k}$

and $\vec{r}(0) = (c_1)\vec{i} + (c_2)\vec{j} - (c_3)\vec{k}$

$= 2\vec{i} - 4\vec{j} + 9\vec{k} \longrightarrow$

$c_1 = 2$, $c_2 = -4$, and $c_3 = -9 \rightarrow$

$$\vec{r}(t) = (t+2)\vec{i} + \left(\tfrac{1}{2}t^2 - 4\right)\vec{j} - \left(\tfrac{1}{3}t^3 - 9\right)\vec{k}$$

2.) $\dfrac{d^2}{dt^2}\vec{r}(t) = \vec{r}\,''(t)$

$$= (\cos t)\vec{i} + (\sin 2t)\vec{j} + (\sin t - \cos t)\vec{k}$$

and $\vec{r}\,'(0) = \vec{i} + \vec{j} - \vec{k}$

and $\vec{r}(0) = \vec{i} - \vec{j} + 3\vec{k}$;

$\xrightarrow{\int} \vec{r}\,'(t) = (\sin t + c_1)\vec{i}$

$$+ \left(-\tfrac{1}{2}\cos 2t + c_2\right)\vec{j} + (-\cos t - \sin t + c_3)\vec{k}$$

and

$$\vec{r}\,'(0) = (\cancel{\sin}^{0}\, 0 + c_1)\vec{i}$$

$$+ \left(-\tfrac{1}{2}\cancel{\cos}_{1}\, 0 + c_2\right)\vec{j} + (-\cancel{\cos}^{1}\, 0 - \cancel{\sin}^{0}\, 0 + c_3)\vec{k}$$

$$= (c_1)\vec{i} + \left(-\tfrac{1}{2} + c_2\right)\vec{j} + (-1 + c_3)\vec{k}$$

$$= (1)\vec{i} + (1)\vec{j} + (-1)\vec{k}$$

$\rightarrow c_1 = 1$, $-\frac{1}{2} + c_2 = 1 \rightarrow c_2 = 3/2$,

$-1+c_3=-1 \rightarrow c_3=0$, so

$$\vec{r}\,'(t)=(\sin t+1)\,\vec{i}+\left(-\tfrac{1}{2}\cos 2t+\tfrac{3}{2}\right)\vec{j}+(-\cos t-\sin t)\,\vec{k}$$

$$\xrightarrow{\int} \vec{r}(t)=(-\cos t+t+c_1)\,\vec{i}+\left(-\tfrac{1}{4}\sin 2t+\tfrac{3}{2}t+c_2\right)\vec{j}+(-\sin t+\cos t+c_3)\,\vec{k}$$

and

$$\vec{r}(0)=(-\overset{1}{\cos 0}+0+c_1)\,\vec{i}+\left(-\tfrac{1}{4}\overset{0}{\sin 0}+0+c_2\right)\vec{j}+(-\overset{0}{\sin 0}+\overset{1}{\cos 0}+c_3)\,\vec{k}$$

$$=(-1+c_1)\,\vec{i}+(c_2)\,\vec{j}+(1+c_3)\,\vec{k}$$

$$=(1)\,\vec{i}+(-1)\,\vec{j}+(3)\,\vec{k}$$

$\rightarrow -1+c_1=1 \rightarrow c_1=2$, $c_2=-1$,

$1+c_3=3 \rightarrow c_3=2$, then

$$\vec{r}(t)=(-\cos t+t+2)\,\vec{i}+\left(-\tfrac{1}{4}\sin 2t+\tfrac{3}{2}t-1\right)\vec{j}+(-\sin t+\cos t+2)\,\vec{k}$$