Math 21D
Kouba
Mappings, Change of Variable, and Jacobians

Jacobian

Def: Let $F(u,v) = (f(u,v), g(u,v)) = (x,y)$ be a mapping from a plane to a plane. The Jacobian of F at point $P = (u_0, v_0)$ is the 2×2 determinant

$$J(P) = \begin{vmatrix} f_u(u_0,v_0) & f_v(u_0,v_0) \\ g_u(u_0,v_0) & g_v(u_0,v_0) \end{vmatrix} .$$

Ex: Let $F(u,v) = (u^2 v, u-v)$, then

$$J(P) = \begin{vmatrix} 2uv & u^2 \\ 1 & -1 \end{vmatrix} = (2uv)(-1) - (u^2)(1)$$

$$= -2uv - u^2, \text{ so}$$

$$J(1,-1) = -2(1)(-1) - (1)^2 = 1$$

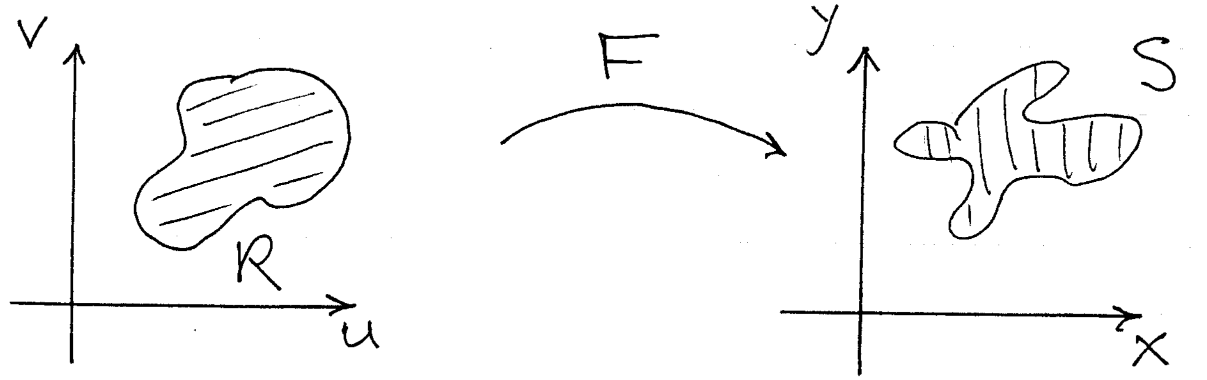
$$J(-2,-3) = -2(-2)(-3) - (-2)^2 = -16$$

Ex: Let $F(u,v) = (au+bv, cu+dv)$, where a, b, c, and d are constants, then F is called a linear map.

Remarks: 1.) F maps lines to lines.

2.) $$J(P) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

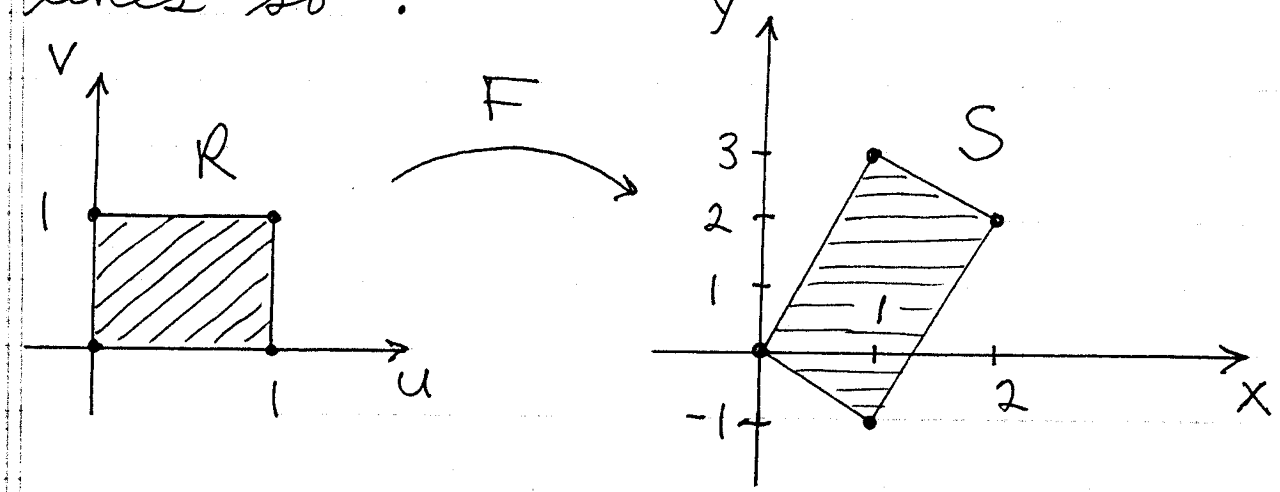
Theorem: Let F be a linear map, let R be a closed region in the uv-plane, and let S be the image of R under F in the xy-plane



Then
$$\text{Area of } S = |ad - bc| \cdot \text{Area of } R.$$

Ex: Let $F(u,v) = (u+v,\ 3u-v) = (x,y)$ and let R be the square in the uv-plane with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$ and its interior. Plot S, the image of R under F, and find the area of S.

$F(0,0)=(0,0)$, $F(1,0)=(1,3)$, $F(0,1)=(1,-1)$, and $F(1,1)=(2,2)$, and F preserves lines so:



$$J(P) = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = (1)(-1) - (1)(3) = -4, \text{ so}$$

$$\text{Area of } S = |-4| \cdot \text{Area of } R$$
$$= (4)(1) = 4$$

<u>Def</u>: We call $|J(P_0)| = |ad-bc|$ the <u>magnification</u> of F at point P_0.

<u>Q</u>: What if F is not a linear map? How do we determine magnification?

<u>A</u>: For $F(u,v) = (f(u,v), g(u,v))$ each of $f(u,v)$ and $g(u,v)$, which represent surfaces in 3D-space, is

approximated with a tangent plane. This leads to the following theorem.

<u>Theorem</u>: Let $F(u,v) = (f(u,v), g(u,v))$ and let $P_0 = (u_0, v_0)$. Then the magnification of F at P_0 is

$$|J(P_0)| = \begin{vmatrix} f_u(P_0) & f_v(P_0) \\ g_u(P_0) & g_v(P_0) \end{vmatrix}.$$

<u>Ex</u>: For $F(u,v) = (u^2 v, u-v)$ we have $J(P) = -2uv - u^2$. Thus, the magnification of F at $P = (1,-1)$ is $|J(P)| = |J(1,-1)| = |1| = 1$, and the magnification of F at $P' = (-2,-3)$ is $|J(P')| = |J(-2,-3)| = |-16| = 16$

<u>Theorem</u>: Let $F(u,v,w) = (f(u,v,w), g(u,v,w), h(u,v,w))$ and let $P_0 = (u_0, v_0, w_0)$. Then the magnification of F at P_0 is

$$|J(P_0)| = \begin{vmatrix} f_u(P_0) & f_v(P_0) & f_w(P_0) \\ g_u(P_0) & g_v(P_0) & g_w(P_0) \\ h_u(P_0) & h_v(P_0) & h_w(P_0) \end{vmatrix}.$$

Ex: Let $F(u,v,w) = (u-v, vw^2, uw) = (x,y,z)$. Find the magnification of F at $(-1,1,-3)$.

$$J(P) = \begin{vmatrix} 1 & -1 & 0 \\ 0 & w^2 & 2vw \\ w & 0 & u \end{vmatrix}$$

$$= (1)\begin{vmatrix} w^2 & 2vw \\ 0 & u \end{vmatrix} - (-1)\begin{vmatrix} 0 & 2vw \\ w & u \end{vmatrix} + (0)\begin{vmatrix} 0 & w^2 \\ w & 0 \end{vmatrix}$$

$= (uw^2 - 0) + (0 - 2vw^2) + 0$

$= uw^2 - 2vw^2$ so magnification of F at $(-1,1,-3)$ is

$|J(-1,1,-3)| = |(-1)(-3)^2 - 2(1)(-3)^2| = |-9-18| = |-27|$

$= 27$.

<u>Change of Variable Theorem</u>: Suppose $F(u,v) = (f(u,v), g(u,v)) = (x,y)$ is a change of variable mapping and $F(P) = Q$. Let S be the image of R under F. Then

$$\iint_S h(Q)\,dA = \iint_S h(Q)\,dy\,dx$$

$$= \iint_R h(F(P)) \cdot |J(P)| \cdot dv\,du.$$

<u>Note</u>: A similar result holds for triple integrals.