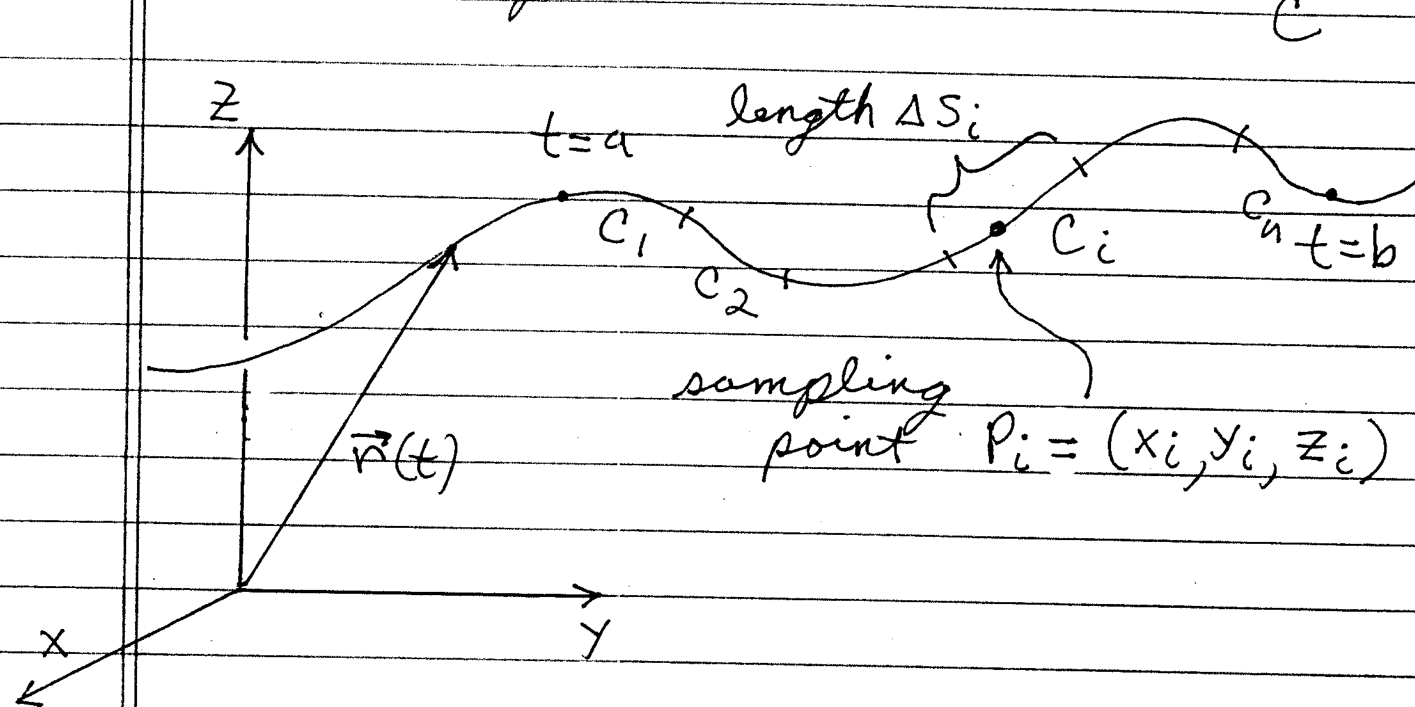
Math 21D
Kouba
Line Integrals



Let C be given by the vector function $\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$ for $t=a$ to $t=b$. Define function $w = f(x,y,z)$ on curve C by $w = f(g(t), h(t), k(t))$. Divide curve C into n parts $C_1, C_2, C_3, \ldots, C_n$ of arc length $\Delta S_1, \Delta S_2, \Delta S_3, \ldots, \Delta S_n$. Pick sampling points $P_i = (x_i, y_i, z_i)$ in C_i for $i = 1, 2, 3, \ldots, n$. We can now define the following new integral.

Def: The line integral of f over curve C from $t=a$ to $t=b$ is

$$\int_C f(x,y,z)\,ds = \lim_{mesh \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \cdot \Delta s_i$$

Method of Evaluation:

If C is determined by $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$ for $a \le t \le b$, then

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t), h(t), k(t)) \cdot \frac{ds}{dt} \cdot dt,$$

where

$$\frac{ds}{dt} = |\vec{v}(t)| = \sqrt{(g'(t))^2 + (h'(t))^2 + (k'(t))^2}.$$

Applications of Line Integrals

Assume that $\delta(x,y,z)$ is the density (mass/length units) at point $P = (x,y,z)$ on curve C determined by $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$ for $t = a$ to $t = b$.

1.) Length $= \int_C 1\,ds$

2.) Mass $= \int_C \delta(P)\,ds$

3.) Moments about planes :

a.) $M_{x=a} = \int_C (x-a)\,\delta(P)\,ds$

b.) $M_{y=a} = \int_C (y-a)\,\delta(P)\,ds$

c.) $M_{z=a} = \int_C (z-a)\,\delta(P)\,ds$

4.) Center of Mass $(\bar{x}, \bar{y}, \bar{z})$:

$$\bar{x} = \frac{\int_C x\,\delta(P)\,ds}{\int_C \delta(P)\,ds}, \quad \bar{y} = \frac{\int_C y\,\delta(P)\,ds}{\int_C \delta(P)\,ds}, \quad \bar{z} = \frac{\int_C z\,\delta(P)\,ds}{\int_C \delta(P)\,ds}$$

5.) Centroid $(\bar{x}, \bar{y}, \bar{z})$:

$$\bar{x} = \frac{\int_C x\,ds}{\int_C 1\,ds}, \quad \bar{y} = \frac{\int_C y\,ds}{\int_C 1\,ds}, \quad \bar{z} = \frac{\int_C z\,ds}{\int_C 1\,ds}$$

6.) Moment of Inertia :

$$\text{M. of I.} = \int_C (\text{distance})^2\,\delta(P)\,ds,$$

where distance refers to the distance from point $P = (x, y, z)$ on curve C to either a point or axis of rotation.