Section 16.1
Thomas Calculus
11th Ed.

Applications of Line Integrals

Example: Consider a thick wire lying along the linear path C determined by the vector function

$$\vec{r}(t) = (t)\vec{i} + (2t)\vec{j} + (2t)\vec{k},$$

for $t = 0$ to $t = 4$. Assume that density at point $P = (x, y, z)$ is given by

$$\delta(P) = \delta(x, y, z) = xy + z \quad \frac{\text{gm.}}{\text{cm.}}$$

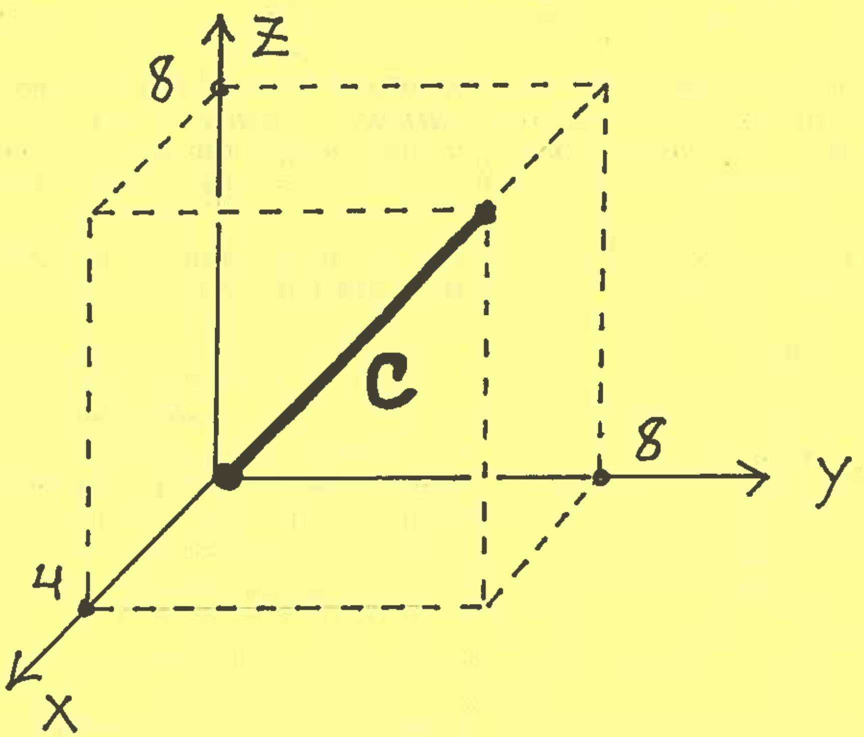
1.) Sketch path C.

2.) Find the total MASS of the wire.

1.)

$$C : \begin{cases} x = t \\ y = 2t \\ z = 2t \end{cases}$$, parametric equations for line in space passing through point (0,0,0) with direction vector

$$\vec{V} = (1)\,\vec{i} + (2)\,\vec{j} + (2)\,\vec{k}$$



Total Mass of wire is

$$\text{MASS} = \int_C \delta(P)\, ds = \int_C (xy+z) \cdot \frac{ds}{dt}\, dt$$

$$= \int_0^4 ((t)(2t) + (2t)) \sqrt{1^2+2^2+2^2}\ dt$$

$$= \int_0^4 (2t^2 + 2t)\sqrt{9}\ dt$$

$$= 3\left(\frac{2}{3}t^3 + t^2\right)\Big|_0^4$$

$$= 2(4)^3 + 3(4)^2 = 176 \text{ gm.}$$

<u>Example</u>: Consider a thick wire lying along path C (helix) determined by the vector function

$$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (t)\vec{k},$$

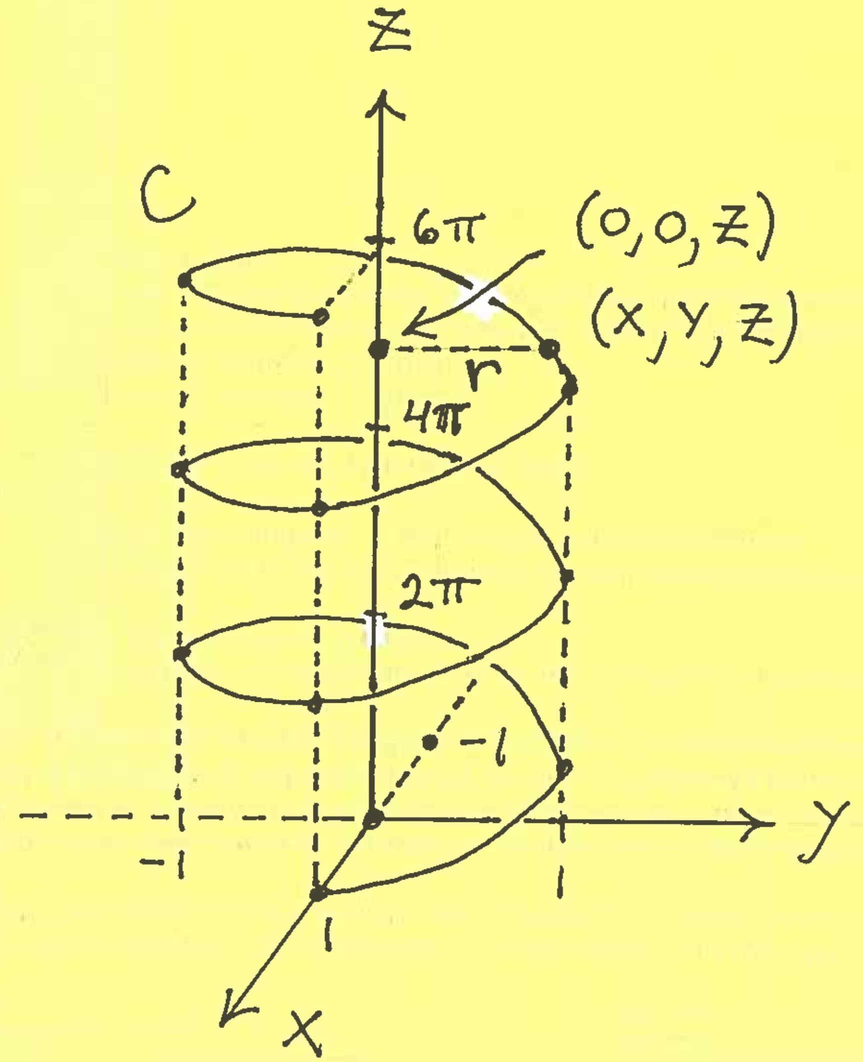
for $t=0$ to $t=6\pi$ (3 loops of the helix). Assume that density at point $P=(x,y,z)$ is given by

$$\delta(P) = \delta(x,y,z) = x^2 + z \quad \text{gm./cm.}$$

1.) Sketch path C.

2.) Find the Moment of Inertia of the wire about the z-axis.

$$C: \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases} \longrightarrow$$



$x^2 + y^2$

$= (\cos t)^2 + (\sin t)^2$

$= 1 \longrightarrow$

$x^2 + y^2 = 1$

(circle)

and

$z = t$:

The distance r from point (x,y,z) to point $(0,0,z)$ on the z-axis is

$$r = \sqrt{(x-0)^2 + (y-0)^2 + (z-z)^2}$$

$$= \sqrt{x^2+y^2} = \sqrt{1} = 1 \text{, i.e.,}$$

$$\boxed{r = 1} \; ;$$

$\vec{v}(t) = (-\sin t)\vec{i} + (\cos t)\vec{j} + (1)\vec{k}$. Then

the Moment of Inertia about the z-axis is

$$\text{M. of I.} = \int_C (\text{distance})^2 \delta(P)\, ds$$

$$= \int_C (1)^2 \cdot (x^2 + z) \cdot \frac{ds}{dt}\, dt$$

$$= \int_0^{6\pi} ((\cos t)^2 + t) \cdot \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt$$

$$= \int_0^{6\pi} (\cos^2 t + t) \sqrt{1+1}\, dt$$

$$= \sqrt{2} \int_0^{6\pi} \left[\frac{1}{2}(1 + \cos 2t) + t \right] dt$$

$$= \sqrt{2} \left[\frac{1}{2} + \frac{1}{2} \cos 2t + t \right] dt$$

$$= \sqrt{2} \left[\frac{1}{2} t + \frac{1}{2} \cdot \frac{1}{2} \sin 2t + \frac{1}{2} t^2 \right] \Big|_0^{6\pi}$$

$$= \sqrt{2} \left[\frac{1}{2}(6\pi) + \frac{1}{4} \sin(12\pi) + \frac{1}{2}(6\pi)^2 \right]$$

$$= \sqrt{2}\left[3\pi + \frac{1}{4}(0) + 18\pi^2\right]$$

$$= \sqrt{2}\left[3\pi + 18\pi^2\right] \quad (\text{gm.})(\text{cm.}^2)$$