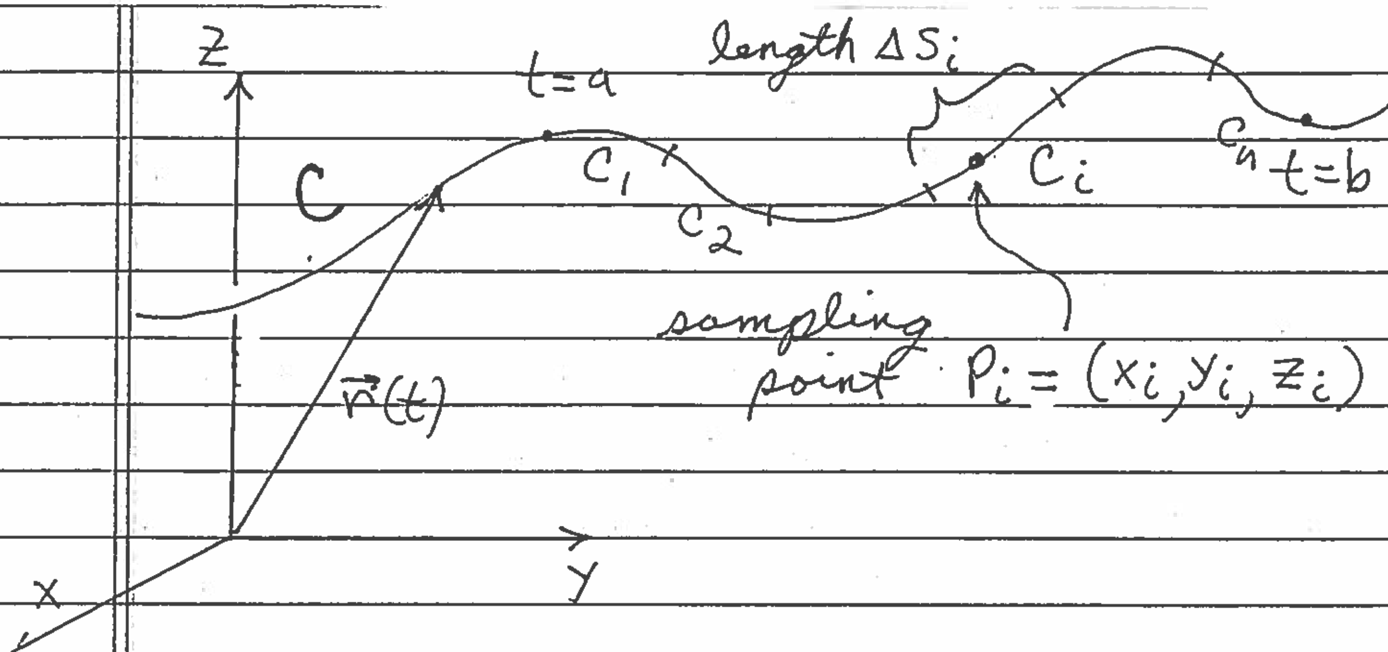
Section 16.1
Thomas Calculus
11th Ed.

Line Integrals Defined on Path C



Let C be given by the vector function $\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$ for $t=a$ to $t=b$. Define function $w = f(x,y,z)$ on curve C by $w = f(g(t), h(t), k(t))$. Divide curve C into n parts $C_1, C_2, C_3, \ldots, C_n$ of arc length $\Delta S_1, \Delta S_2, \Delta S_3, \ldots, \Delta S_n$. Pick sampling points $P_i = (x_i, y_i, z_i)$ in C_i for $i = 1, 2, 3, \ldots, n$. We can now define the following new integral.

Def: The line integral of f over curve C from $t=a$ to $t=b$ is

$$\int_C f(x,y,z)\,ds = \lim_{mesh \to 0} \sum_{i=1}^{n} f(x_i,y_i,z_i)\cdot \Delta s_i$$

<u>Method of Evaluation</u>:
If C is determined by
$\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$ for
$a \le t \le b$, then

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t),h(t),k(t))\cdot \frac{ds}{dt}\cdot dt,$$

where
$$\frac{ds}{dt} = |\vec{v}(t)| = \sqrt{(g'(t))^2 + (h'(t))^2 + (k'(t))^2}\,.$$

<u>Applications of Line Integrals</u>

Assume that $\delta(x,y,z)$ is the density (mass/length units) at point $P = (x,y,z)$ on curve C determined by $\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$ for $t = a$ to $t = b$.

1.) Length $= \int_C 1\,ds$

2.) Mass $= \int_C \delta(P)\,ds$

3.) Moments about planes :

a.) $M_{x=a} = \int_C (x-a)\,\delta(P)\,ds$

b.) $M_{y=a} = \int_C (y-a)\,\delta(P)\,ds$

c.) $M_{z=a} = \int_C (z-a)\,\delta(P)\,ds$

4.) Center of Mass $(\bar{x}, \bar{y}, \bar{z})$:

$$\bar{x} = \frac{\int_C x\,\delta(P)\,ds}{\int_C \delta(P)\,ds}, \quad \bar{y} = \frac{\int_C y\,\delta(P)\,ds}{\int_C \delta(P)\,ds}, \quad \bar{z} = \frac{\int_C z\,\delta(P)\,ds}{\int_C \delta(P)\,ds}$$

5.) Centroid $(\bar{x}, \bar{y}, \bar{z})$:

$$\bar{x} = \frac{\int_C x\,ds}{\int_C 1\,ds}, \quad \bar{y} = \frac{\int_C y\,ds}{\int_C 1\,ds}, \quad \bar{z} = \frac{\int_C z\,ds}{\int_C 1\,ds}$$

6.) Moment of Inertia :

$$\text{M. of I.} = \int_C (\text{distance})^2\,\delta(P)\,ds,$$

where distance refers to the distance from point $P=(x,y,z)$ on curve C to either a point or axis of rotation.

Example: Consider path C determined by the vector function $\vec{r}(t) = \left(\frac{3}{2}t^2\right)\vec{i} + (t^3)\vec{j}$ for $t=0$ to $t=\sqrt{15}$. Evaluate the line integral $\int_C 1\, ds$.

$\vec{v}(t) = (3t)\vec{i} + (3t^2)\vec{j}$, so

$$\int_C 1\, ds = \int_0^{\sqrt{15}} \frac{ds}{dt}\, dt$$

$$= \int_0^{\sqrt{15}} \sqrt{(3t)^2 + (3t^2)^2}\, dt$$

$$= \int_0^{\sqrt{15}} \sqrt{9t^2 + 9t^4}\, dt$$

$$= \int_0^{\sqrt{15}} \sqrt{9t^2(1+t^2)}\, dt$$

$$= \int_0^{\sqrt{15}} 3t\sqrt{1+t^2}\, dt$$

Example : Consider path C determined by the vector function $\vec{r}(t) = (\sqrt{t})\vec{i} + (t)\vec{j} + (\frac{1}{2}t)\vec{k}$ for $t=3$ to $t=7$. Evaluate the line integral $\int_C \sqrt{y}\, ds$.

$\vec{v}(t) = \left(\frac{1}{2\sqrt{t}}\right)\vec{i} + (1)\vec{j} + \left(\frac{1}{2}\right)\vec{k}$, so

$$\int_C \sqrt{y}\, ds = \int_3^7 \sqrt{t} \cdot \frac{ds}{dt}\, dt$$

$$= \int_3^7 \sqrt{t} \sqrt{\left(\frac{1}{2\sqrt{t}}\right)^2 + (1)^2 + \left(\frac{1}{2}\right)^2}\, dt$$

$$= \int_3^7 \sqrt{t} \sqrt{\frac{1}{4t} + 1 + \frac{1}{4}}\, dt$$

$$= \int_3^7 \sqrt{t} \sqrt{\frac{1}{4t} + \frac{4t}{4t} + \frac{t}{4t}}\, dt$$

$$= \int_3^7 \sqrt{t} \sqrt{\frac{1+5t}{4t}}\, dt$$

$$= \int_3^7 \cancel{\sqrt{t}} \cdot \frac{\sqrt{1+5t}}{2\cancel{\sqrt{t}}}\, dt$$

$$= \frac{1}{2} \int_3^7 \sqrt{1+5t}\, dt$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} \cdot \frac{1}{5} (1+5t)^{3/2} \Big|_3^7$$

$$= \frac{1}{15}(36)^{3/2} - \frac{1}{15}(16)^{3/2}$$

$$= \frac{1}{15}(6^3) - \frac{1}{15}(4^3)$$

$$= \frac{152}{15} .$$

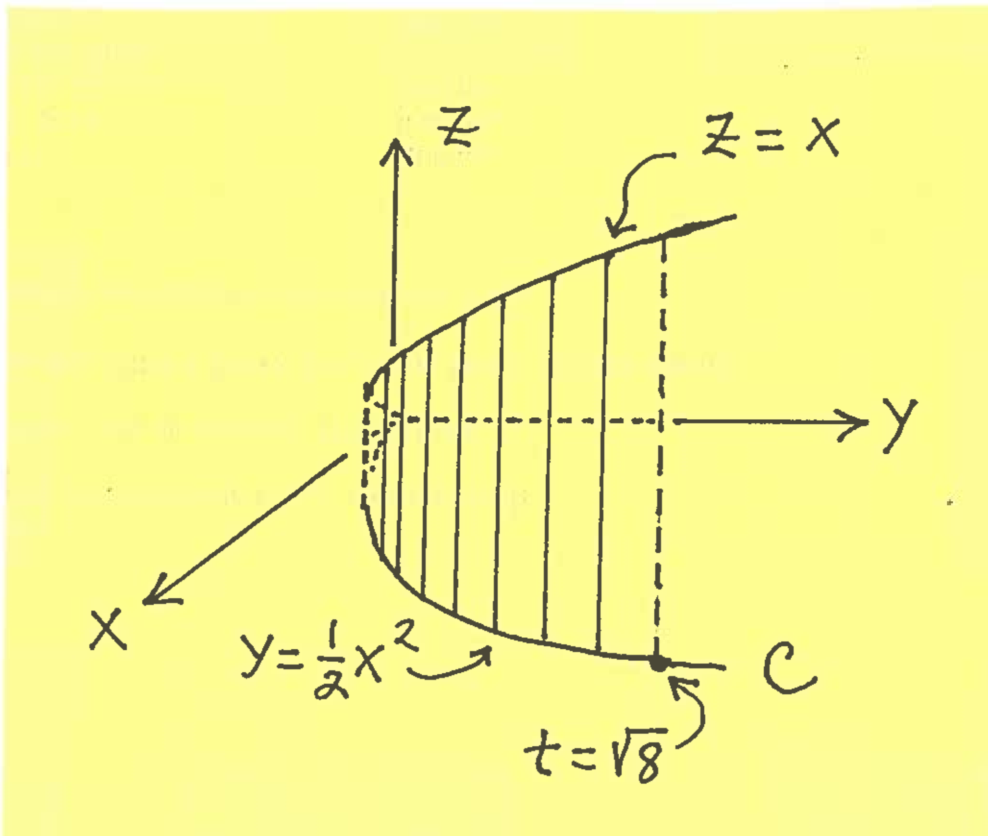
<u>Example</u>: Consider path C determined by vector function $\vec{r}(t) = (t)\vec{i} + (\frac{1}{2}t^2)\vec{j}$. Consider a wall of height x cm. directly above path C.

1.) Sketch path C and the wall.

2.) Find the AREA of the wall

from $t=0$ to $t=\sqrt{8}$.

1.) $C: \begin{cases} x=t \\ y=\frac{1}{2}t^2 \end{cases} \rightarrow y=\frac{1}{2}x^2$



2.) $\vec{v}(t) = (1)\vec{i} + (t)\vec{j}$, so AREA of wall is

$$\text{AREA} = \int_C x\, ds = \int_0^{\sqrt{8}} t \cdot \frac{ds}{dt}\, dt$$

$$= \int_0^{\sqrt{8}} t \sqrt{1^2 + t^2}\, dt$$

$$= \int_0^{\sqrt{8}} t \sqrt{1 + t^2}\, dt$$

$$= \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}} (1 + t^2)^{3/2} \Big|_0^{\sqrt{8}}$$

$$= \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2}$$

$$= \frac{1}{3}(27) - \frac{1}{3}(1)$$

$$= \frac{26}{3} \text{ cm.}^2$$