Section 15.7
Thomas Calculus
11th Ed.

Mappings, Change of Variable, and Jacobians

Mappings (Functions)

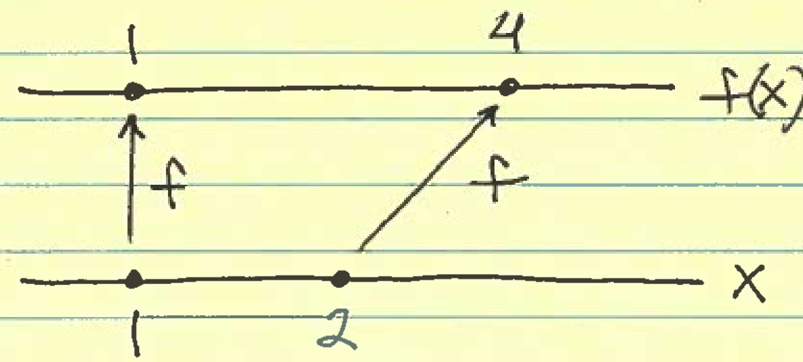
I.) Line to Line:

Example:

$f(x) = x^2$;

$f(1) = 1$,

$f(2) = 4$



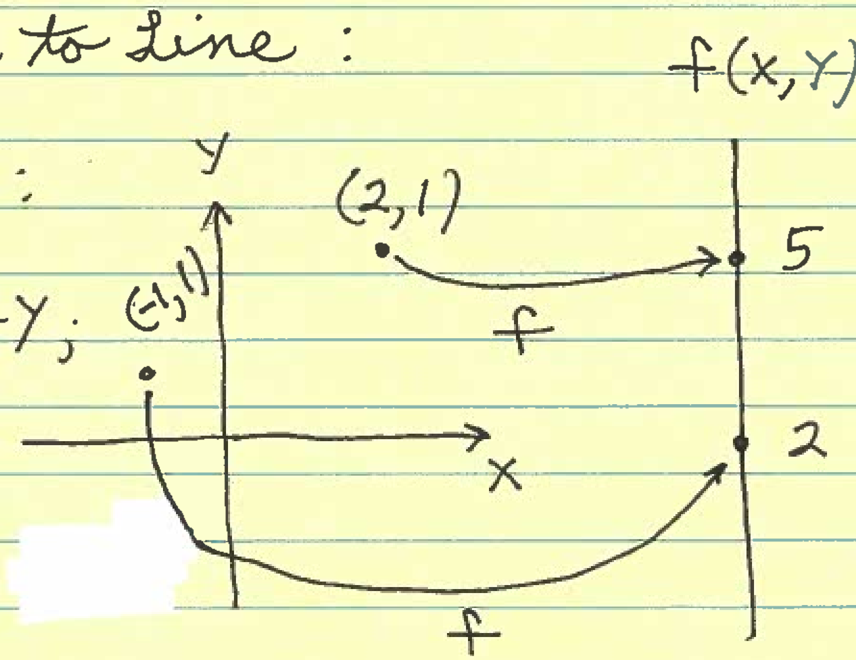
II.) Plane to Line:

Example:

$f(x,y) = x^2 + y$;

$f(2,1) = 4 + 1 = 5$,

$f(-1,1) = 1 + 1 = 2$



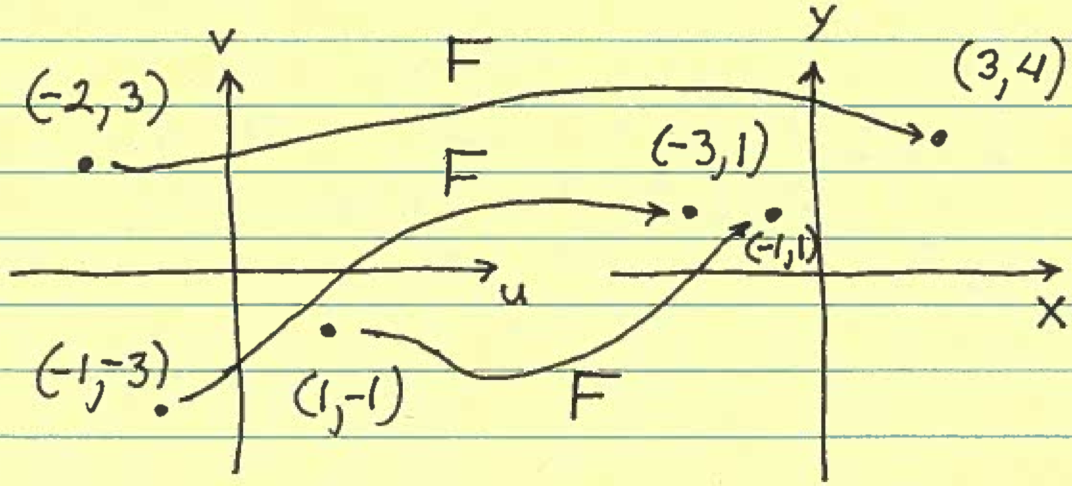
III.) Plane to Plane:

Example: $F(u,v) = (v, u^2) = (x, y)$

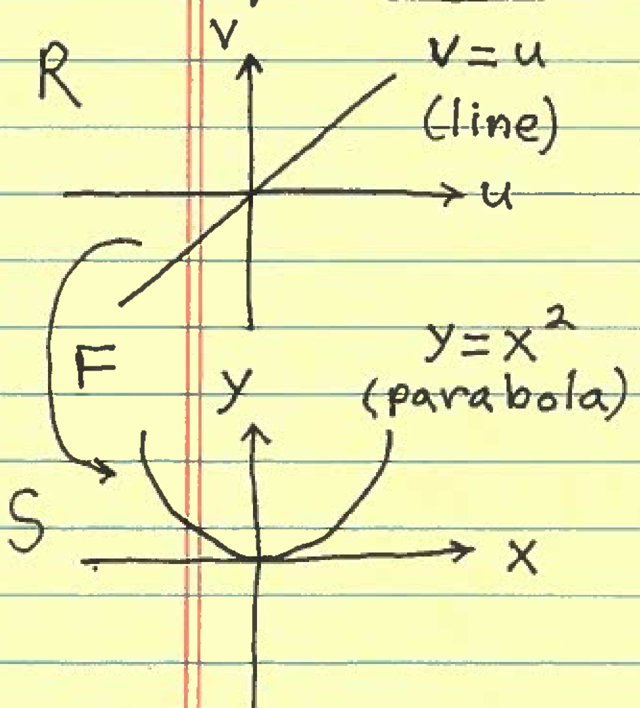
$F(-2,3) = (3,4)$,

$F(-1,-3) = (-3,1)$,

$F(1,-1) = (-1,1)$



Let R be the set of all points (u,v) on the line $v = u$ in the uv-plane. Find the image S of R under mapping F in the xy-plane:



$$\left.\begin{array}{l} x = v \\ y = u^2 \end{array}\right\} \text{ and } v = u \rightarrow$$

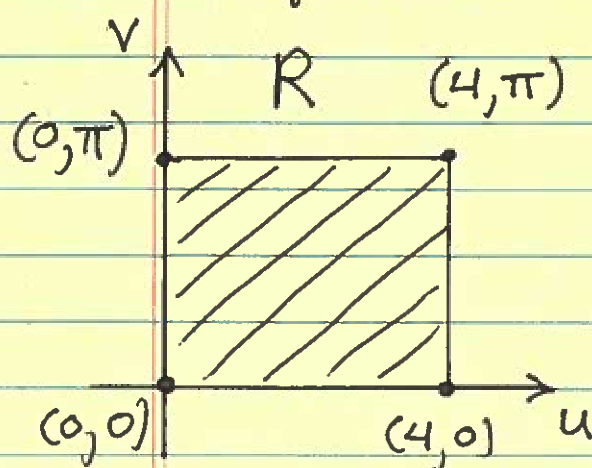
$$\left.\begin{array}{l} x = u \\ y = u^2 \end{array}\right\} \rightarrow y = x^2$$

(parabola)

Example: Consider the Plane-to-Plane mapping given by

$$F(u,v) = (u\cos v,\ u\sin v) = (x,y)\ .$$

Let region R be the given rectangle in the uv-plane. Find and sketch the image S of R under mapping F in the xy-plane:



$0 \le u \le 4$,
$0 \le v \le \pi$;

Let's first consider the mappings of the corners;

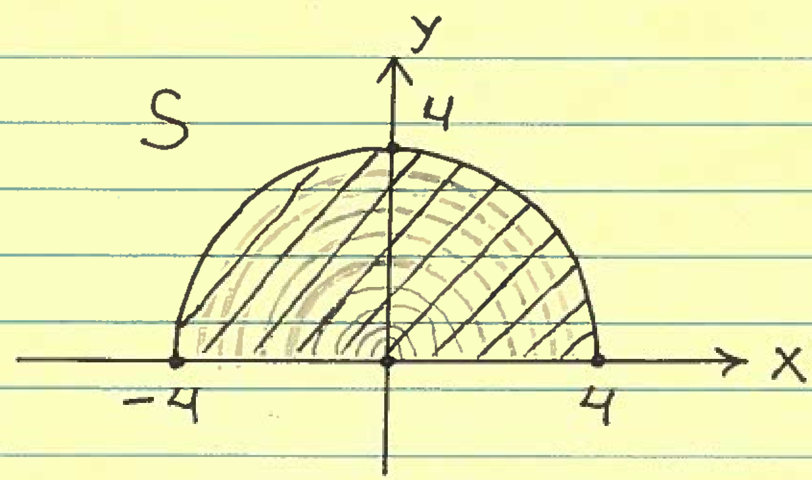
$F(0,0) = (0,0)$,
$F(4,0) = (4,0)$,
$F(4,\pi) = (-4,0)$,
$F(0,\pi) = (0,0)$;

Then $\left.\begin{array}{l} x = u\cos v \\ y = u\sin v \end{array}\right\} \rightarrow$

$$x^2 + y^2 = u^2\cos^2 v + u^2\sin^2 v$$
$$= u^2(\cos^2 v + \sin^2 v) = u^2(1) = u^2,$$

i.e., $\boxed{x^2+y^2=u^2}$,

a <u>circle</u> <u>of</u> <u>radius</u> <u>u</u>, $0 \le u \le 4$;
BUT $0 \le v \le \pi$ and $y = u \sin v$,
so $y \ge 0$, so only use TOP
<u>semi-circles</u>:



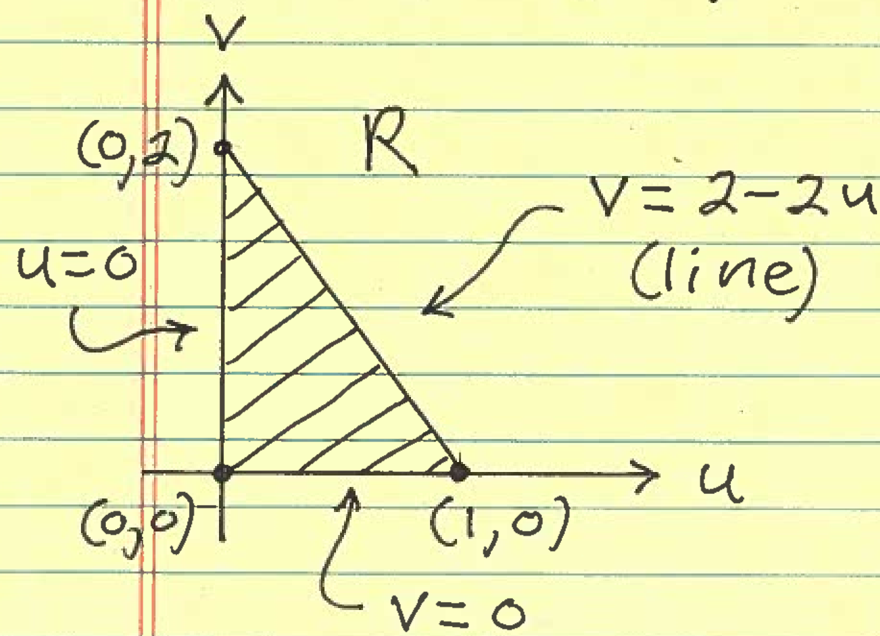
S: all points (x,y) on or inside
the semi-circle $y=\sqrt{16-x^2}$
and <u>above the line</u> $y=0$.

<u>Example</u>: Consider the

Plane-to-Plane mapping
given by

$$F(u,v) = (u-v, u+v) = (x,y) .$$

Let region R be the given rectangle in the uv-plane. Find and sketch the <u>image</u> S of R <u>under</u> <u>mapping</u> F in the xy-plane:



Let's first consider the mappings of the corners;

$F(0,0) = (0,0)$,
$F(0,2) = (-2,2)$,
$F(1,0) = (1,1)$;

$0 \le u \le 1$,
$0 \le v \le 2$;

Now let's consider the mappings of the three edges of region R:

(edge $u=0$): $F(0,v) = (-v,v) = (x,y)$

$$\rightarrow \begin{cases} x = -v \\ y = v \end{cases} \rightarrow \boxed{y = -x} \;;$$

(edge $v=0$): $F(u,0)=(u,u)=(X,Y)$

$$\rightarrow \begin{cases} x=u \\ y=u \end{cases} \longrightarrow \boxed{Y=X} \quad ;$$

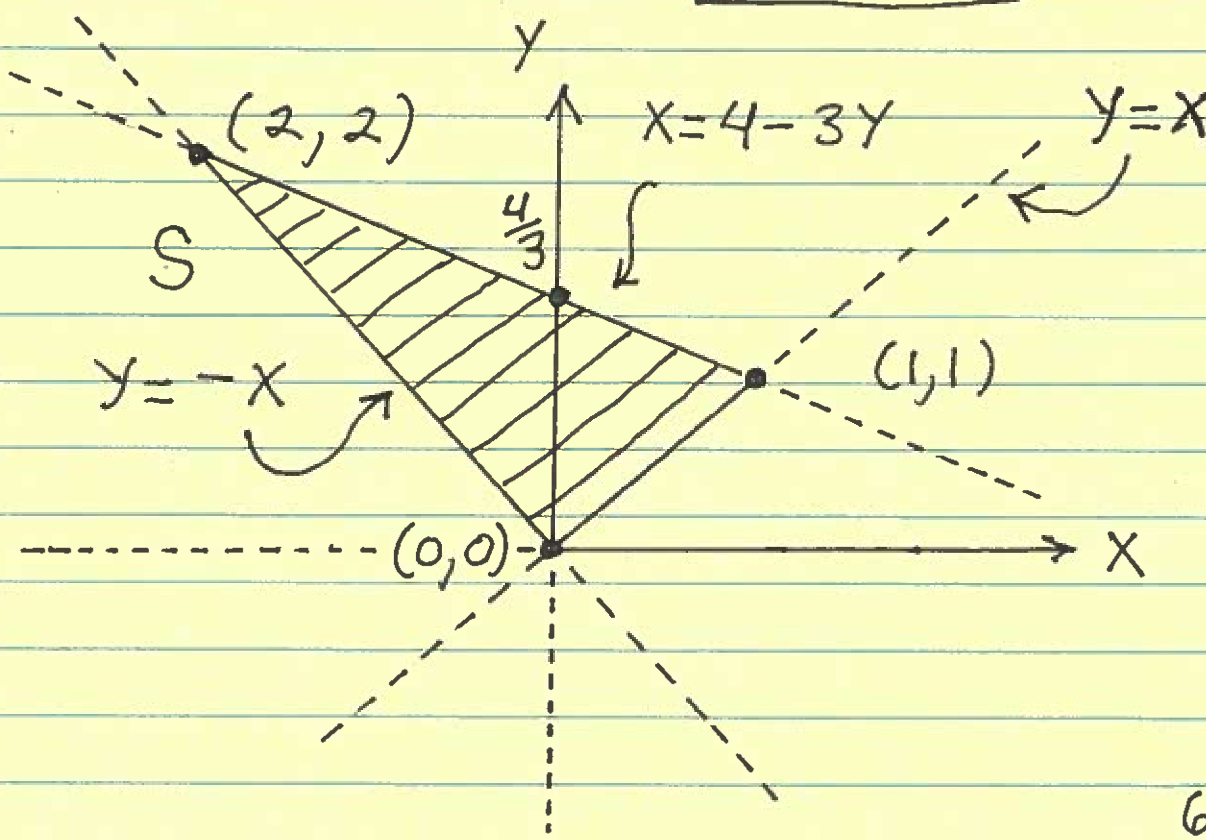
(edge $v=2-2u$):

$$F(u,2-2u)=(u-(2-2u),\ u+(2-2u))$$

$$=(3u-2,\ 2-u)=(X,Y)$$

$$\rightarrow \begin{cases} X=3u-2 \\ Y=2-u \rightarrow u=2-Y \end{cases} \longrightarrow X=3(2-Y)-2$$

$$\rightarrow X=6-3Y-2 \longrightarrow \boxed{X=4-3Y}$$



<u>Example</u> : (a bit challenging)
Consider the mapping F given in the previous example and the corresponding region R (in the uv-plane) and region S (in the xy-plane).

a.) For each point (x,y) in S find the point (u,v) in R which maps to (x,y) under mapping F. b.) Then find the point (u,v) in R which maps to point $(-1, 4/3)$ in S.

$$F(u,v) = (u-v, u+v) = (x,y)$$

$$\rightarrow \begin{cases} x = u-v \\ y = u+v \end{cases} \rightarrow$$

$$x+y = (u-v)+(u+v) = 2u \rightarrow$$

$$\boxed{u = \frac{1}{2}x + \frac{1}{2}y} \;;$$

$$y-x = (u+v)-(u-v) = 2v \longrightarrow$$

$$\boxed{v = \frac{1}{2}y - \frac{1}{2}x}$$

Define mapping $G: S \to R$ by

a.) $G(x,y) = \left(\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}y - \frac{1}{2}x\right) = (u, v)$

b.) $G(-1, 4/3) = \left(\frac{1}{2}(-1) + \frac{1}{2}\left(\frac{4}{3}\right), \frac{1}{2}\left(\frac{4}{3}\right) - \frac{1}{2}(-1)\right)$

$= \left(-\frac{1}{2} + \frac{2}{3}, \frac{2}{3} + \frac{1}{2}\right)$

$= \left(\frac{1}{6}, \frac{7}{6}\right)$