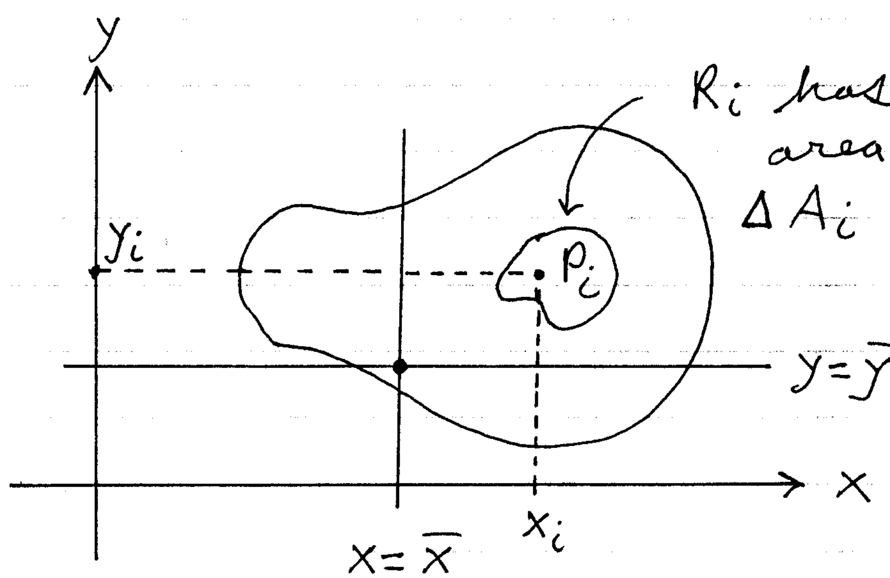
Math 21D
Kouba
Moments, Center of Mass, Centroid, and Moment of Inertia in 2D-Space

Consider a flat region R with variable density $\left(\frac{\text{mass}}{\text{area}} \text{ units}\right)$ $\delta(P)$ at point P. Partition region R into n parts $R_1, R_2, R_3, \ldots, R_n$ of areas $\Delta A_1, \Delta A_2, \Delta A_3, \ldots, \Delta A_n$, resp. Pick sampling point

$$P_i = (x_i, y_i) \text{ in } R_i \text{ for } i = 1, 2, 3, \ldots, n.$$

Consider the vertical line $x = \bar{x}$. The moment of R_i about line $x = \bar{x}$ is



$$\begin{aligned} M_i &= (\text{mass})(\text{distance}) \\ &= (\text{area})(\text{density})(\text{distance}) \\ &= (\Delta A_i)(\delta(P_i))(x_i - \bar{x}) \end{aligned}$$

The total moment of region R

about line $x = \bar{x}$ is

$$M_{x=\bar{x}} = \lim_{mesh \to 0} \sum_{i=1}^{n} \delta(P_i) \cdot (x_i - \bar{x}) \cdot \Delta A_i \to$$

$$\boxed{M_{x=\bar{x}} = \iint_R \delta(P)\,(x - \bar{x})\,dA}\,.$$

at the center of mass of R we have $M_{x=\bar{x}} = 0 \to$

$$\iint_R \delta(P)\,(x-\bar{x})\,dA = 0 \to$$

$$\iint_R x \cdot \delta(P)\,dA - \iint_R \bar{x} \cdot \delta(P)\,dA = 0 \to$$

$$\iint_R x \cdot \delta(P)\,dA = \bar{x} \iint_R \delta(P)\,dA \to$$

$$\boxed{\bar{x} = \frac{\iint_R x \cdot \delta(P)\,dA}{\iint_R \delta(P)\,dA}}\,.$$

Consider the horizontal line $y = \bar{y}$. In an analagous fashion, the total moment of region R about line $y = \bar{y}$ is

$$\boxed{M_{y=\bar{y}} = \iint_R \delta(P)\,(y - \bar{y})\,dA}\,.$$

at the center of mass of R we have

$M_{y=\bar{y}} = 0 \longrightarrow$

$$\boxed{\bar{y} = \frac{\iint_R y\, \delta(P)\, dA}{\iint_R \delta(P)\, dA}}\ .$$

Note: $\iint_R \delta(P)\, dA$ = total mass of R

Note: If the density at the point P is $\delta(P) = k$, a constant, then the center of mass is called the centroid (geometric center) of region R and is given by

$$\boxed{\bar{x} = \frac{\iint_R x\, dA}{\iint_R 1\, dA}}$$

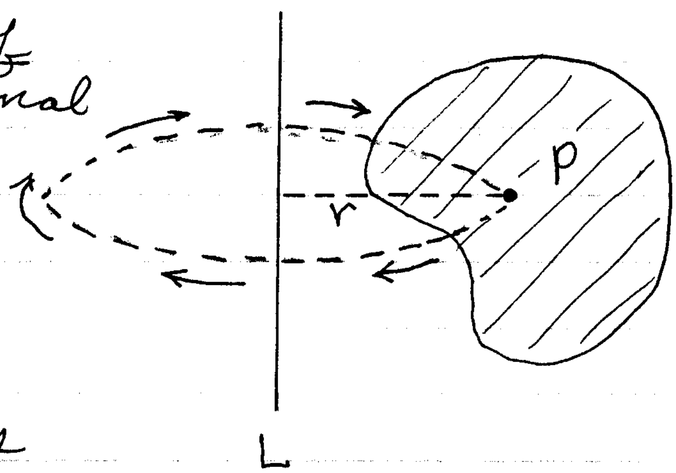
and

$$\boxed{\bar{y} = \frac{\iint_R y\, dA}{\iint_R 1\, dA}}\ .$$

__Note__ : $\iint_R 1 \, dA$ = total area of R

Consider a flat region R with density $\delta(P)$ at point P and which is rotating around a line L or fixed point P_0.

The __moment of inertia__ (rotational inertia) is a measure of the object's resistance to changes in its rotation. It depends on the object's total mass and how far each bit of mass is from the axis or point of rotation.



__Def__: The moment of inertia of R is

$$\boxed{\text{M. of I.} = \iint_R r^2 \, \delta(P) \, dA},$$

where r is the distance from P to the axis or point of rotation.