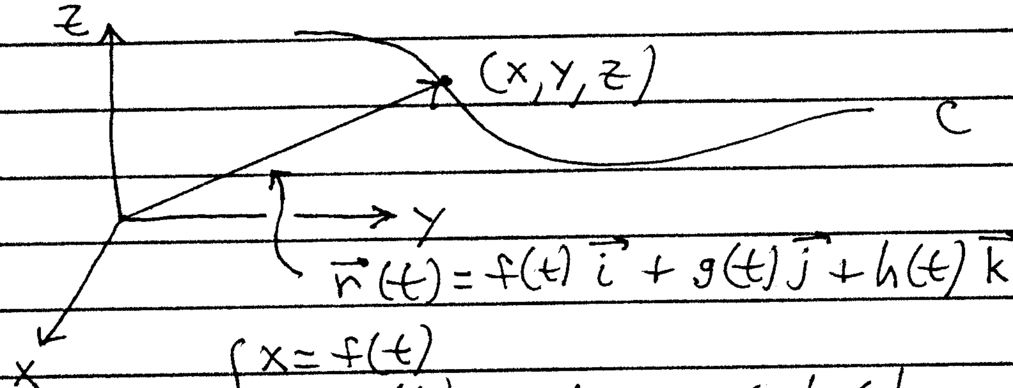
Math 21D
Kouba
Parametrized Surfaces

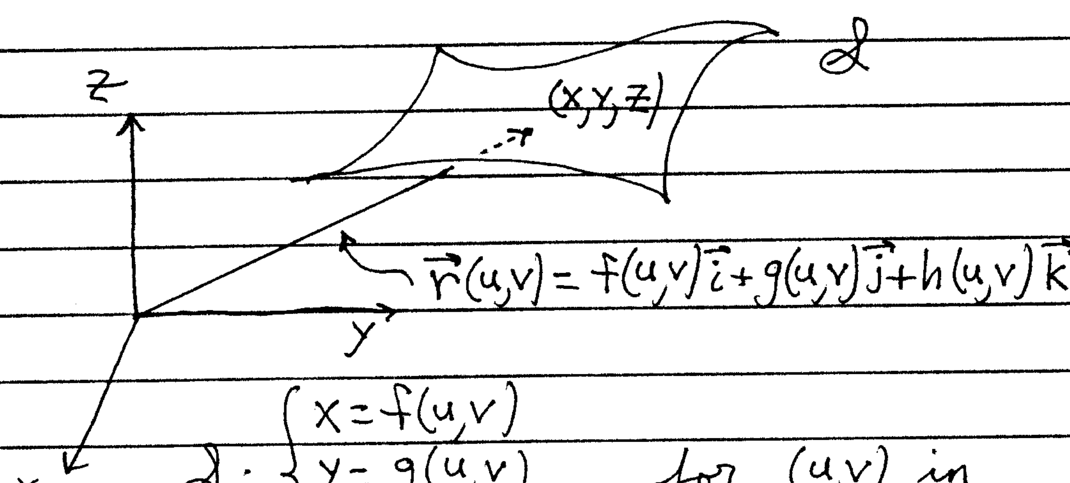
Recall: (Parametrized Path C in 3D-Space)



$$\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

$$C = \begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases} \quad \text{for } a \le t \le b$$

We want to generalize this parametrization to surfaces in 3D-space:



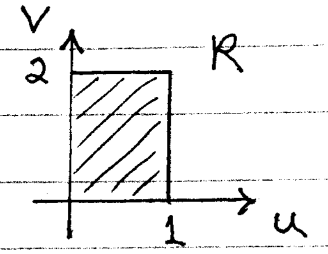
$$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$$

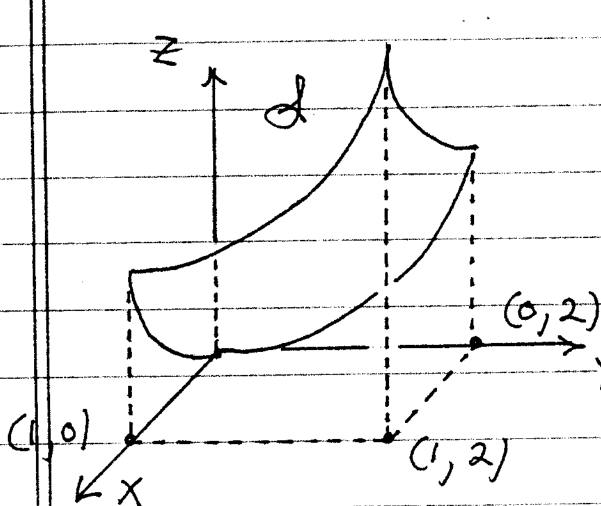
$$S: \begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \end{cases} \quad \text{for } (u,v) \text{ in region } R,$$

e.g., $a \le u \le b$, $l(u) \le v \le k(u)$.

<u>Ex</u>: Parametrize the following surfaces.

1.) S: portion of paraboloid $z = x^2 + y^2$ above rectangle with vertices $(0,0)$, $(1,0)$, $(0,2)$, and $(1,2)$:

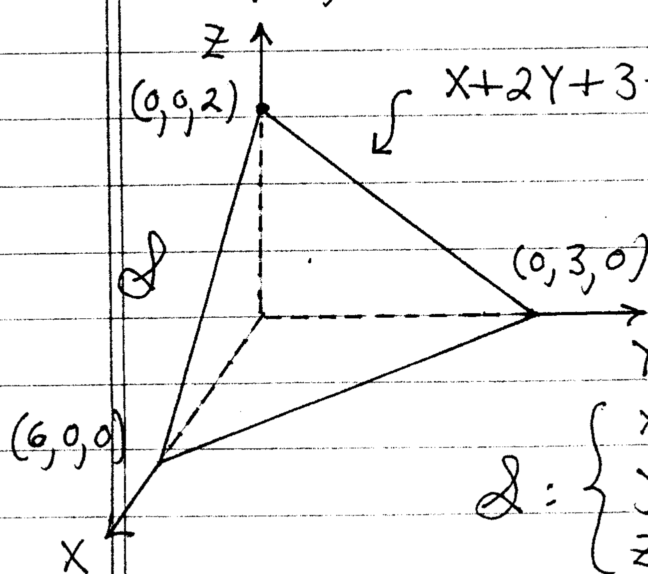


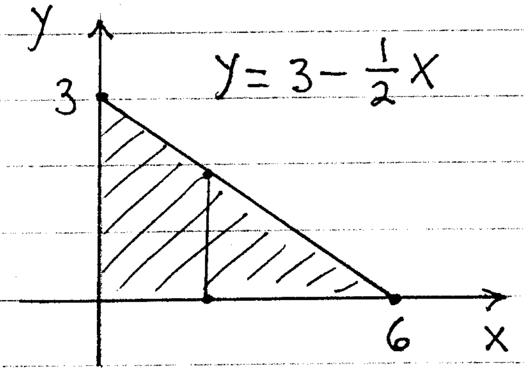


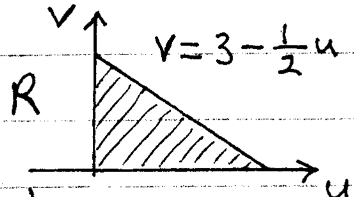
$$S: \begin{cases} x = u \\ y = v \\ z = u^2 + v^2 \end{cases}$$

for $0 \le u \le 1$, $0 \le v \le 2$

2.) S: triangle with vertices $(6,0,0)$, $(0,3,0)$, and $(0,0,2)$:



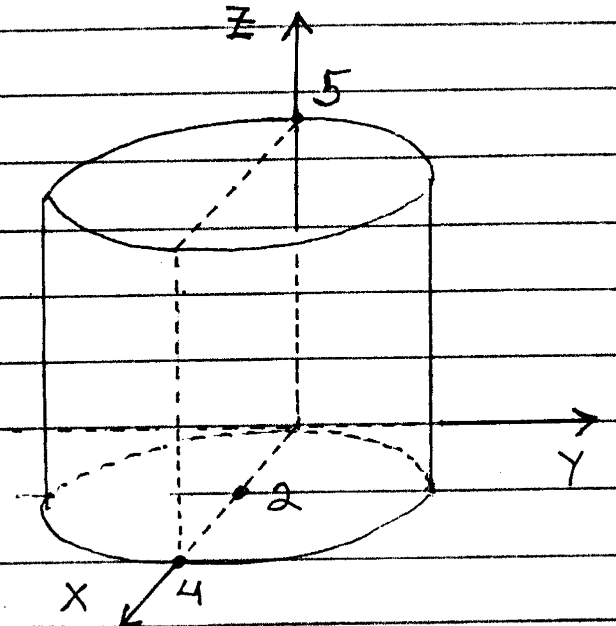


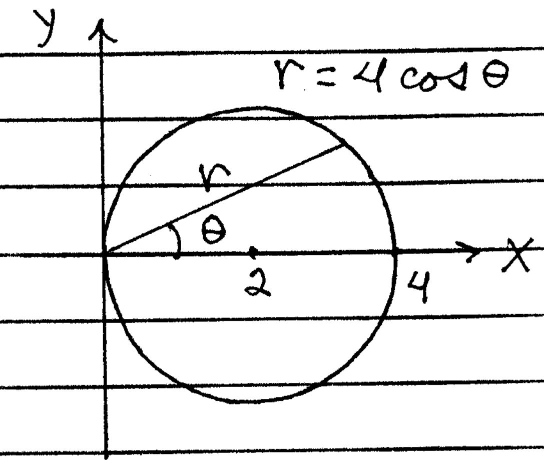


$$S: \begin{cases} x = u \\ y = v \\ z = \frac{1}{3}(6 - x - 2y) = \frac{1}{3}(6 - u - 2v) \end{cases}$$

for $0 \le u \le 6$, $0 \le v \le 3 - \frac{1}{2}u$

3.) S : surface of cylinder $(x-2)^2 + y^2 = 4$ cut by the planes $z = 0$ and $z = 5$:

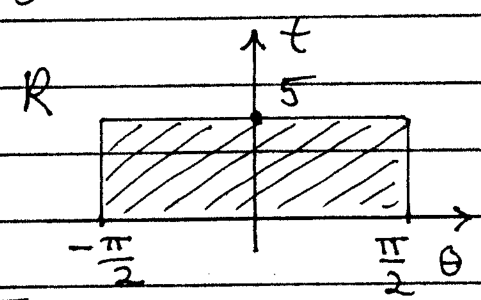




(Recall : (polar coordinates)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$) then

$$S : \begin{cases} x = r\cos\theta = (4\cos\theta)\cos\theta \\ y = r\sin\theta = (4\cos\theta)\sin\theta \\ z = t \end{cases} \rightarrow$$

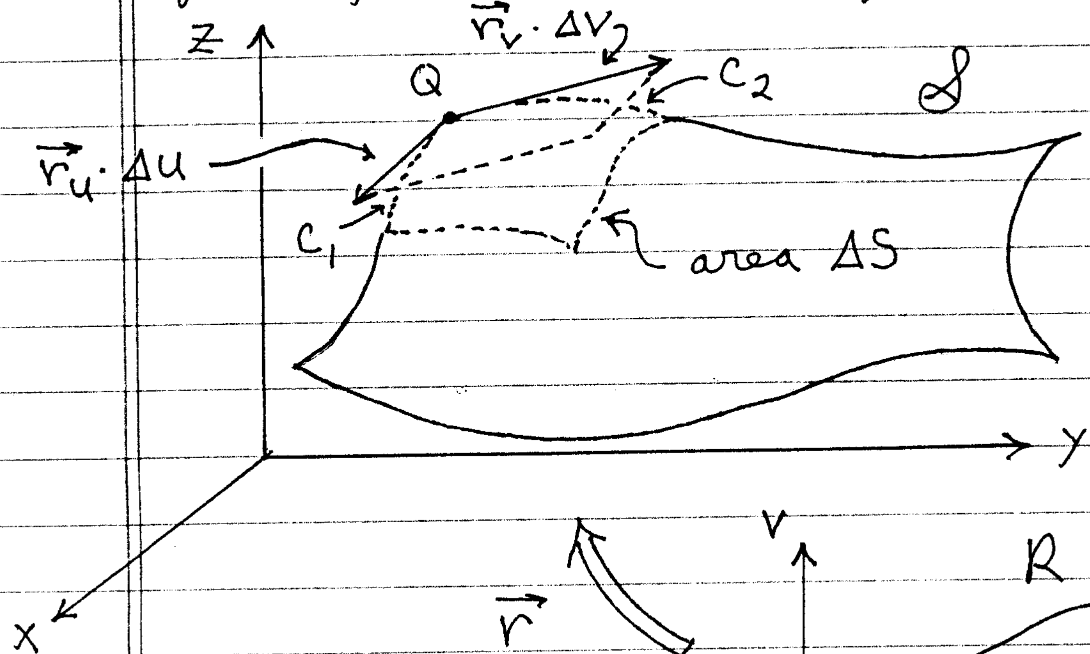


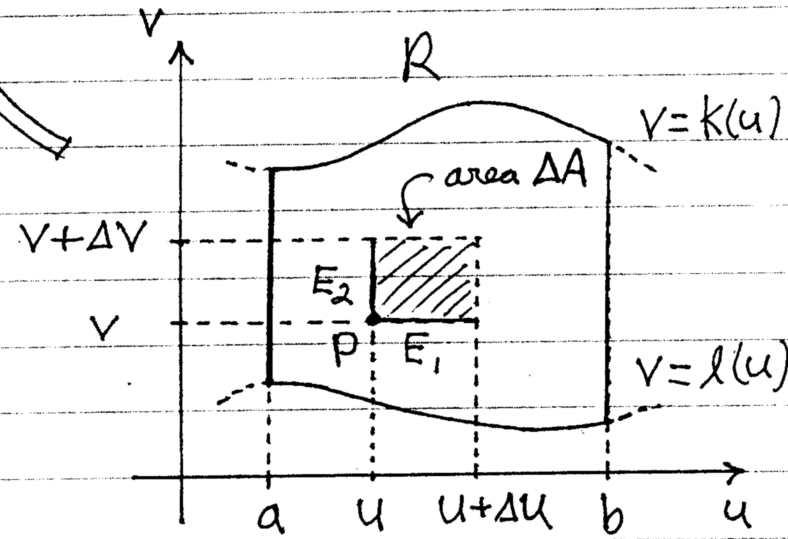
$$S : \begin{cases} x = 4\cos^2\theta \\ y = 4\sin\theta\cos\theta \\ z = t \end{cases}$$

for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $0 \leq t \leq 5$

Finding Area of Parametrized Surface

Let $\mathcal{S}$ be surface parametrized by
$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$
for (u,v) in R : $a \leq u \leq b$, $l(u) \leq v \leq k(u)$





assume that
$\vec{r}(P) = Q$;
let path C_1 be
the image of
edge E_1 under
$\vec{r}$ and let C_2
be the image of edge E_2 under $\vec{r}$;

(<u>Recall</u> : If $y = f(x)$, then $\Delta f \approx f'(x) \cdot \Delta x$.)

then

$$\text{path } C_1 \approx \Delta\vec{r} \text{ (along } E_1) \approx \vec{r}_u(Q)\cdot\Delta u$$

and

$$\text{path } C_2 \approx \Delta\vec{r} \text{ (along } E_2) \approx \vec{r}_v(Q)\cdot\Delta v \quad ;$$

Recall: area of ▱ formed by vectors $\vec{r}_u\cdot\Delta u$ and $\vec{r}_v\cdot\Delta v$ is (magnitude of cross product)

$$|\vec{r}_u\cdot\Delta u \times \vec{r}_v\cdot\Delta v| = |\vec{r}_u\times\vec{r}_v|\cdot\underbrace{\Delta v\cdot\Delta u}_{\Delta A} \quad ;$$

it follows that

$$\text{area } \Delta S \approx |\vec{r}_u\times\vec{r}_v|\cdot\Delta A \text{ , so that} \ldots$$

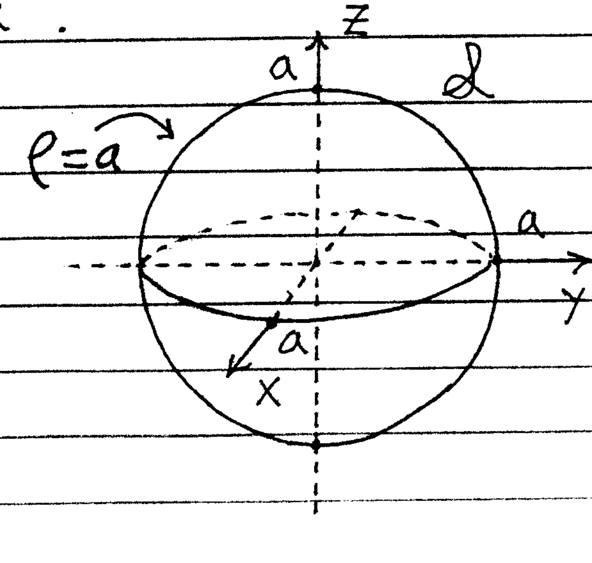
Def: If surface $\mathcal{S}$ is given by

$$\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$$

for (u,v) in R: $a \le u \le b$, $l(u) \le v \le k(u)$, then

$$\text{Area } \mathcal{S} = \iint_{\mathcal{S}} 1\, dS = \iint_R |\vec{r}_u \times \vec{r}_v|\, dA$$

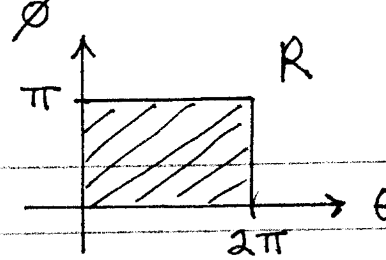
Ex: Find the surface area of a sphere of radius a.



(Recall: (spherical coordinates)

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

)



then

$$S : \begin{cases} x = a \sin\phi \cos\theta \\ y = a \sin\phi \sin\theta \\ z = a \cos\phi \end{cases}$$

for $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$; now

$S : \vec{r}(\phi, \theta)$ and

$$\begin{cases} \vec{r}_\theta = x_\theta \cdot \vec{i} + y_\theta \cdot \vec{j} + z_\theta \cdot \vec{k} \\ \quad = (-a \sin\phi \sin\theta)\vec{i} + (a \sin\phi \cos\theta)\vec{j} + (0)\vec{k}, \\ \vec{r}_\phi = x_\phi \cdot \vec{i} + y_\phi \cdot \vec{j} + z_\phi \cdot \vec{k} \\ \quad = (a\cos\phi\cos\theta)\vec{i} + (a\cos\phi\sin\theta)\vec{j} + (-a\sin\phi)\vec{k} : \end{cases}$$

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \end{vmatrix}$$

$$\begin{aligned} &= (-a^2\sin^2\phi\cos\theta - 0)\ \vec{i} \\ &\quad - (a^2\sin^2\phi\sin\theta - 0)\ \vec{j} \\ &\quad + (-a^2\sin\phi\cos\phi\sin^2\theta - a^2\sin\phi\cos\phi\cos^2\theta)\ \vec{k} \end{aligned}$$

$$\begin{aligned} &= (-a^2\sin^2\phi\cos\theta)\ \vec{i} \\ &\quad + (-a^2\sin^2\phi\sin\theta)\ \vec{j} \\ &\quad + (-a^2\sin\phi\cos\phi\underbrace{(\sin^2\theta + \cos^2\theta)}_{1})\ \vec{k}, \end{aligned}$$

so that the magnitude of $|\vec{r}_\theta \times \vec{r}_\phi|$ is

$$|\vec{r}_\theta \times \vec{r}_\phi| = \sqrt{(-a^2\sin^2\phi\cos\theta)^2 + (-a^2\sin^2\phi\sin\theta)^2 + (-a^2\sin\phi\cos\phi)^2}$$

$$= \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{a^4\sin^4\phi\,(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + a^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{a^4\sin^2\phi\,(\underbrace{\sin^2\phi + \cos^2\phi}_{1})}$$

$$= a^2\sin\phi \quad ; \quad \text{then}$$

$$\text{Area } \mathcal{S} = \iint_{\mathcal{S}} 1\,dS = \iint_R |\vec{r}_\theta \times \vec{r}_\phi|\,dA$$

$$= \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\left(-a^2\cos\phi\,\Big|_{\phi=0}^{\phi=\pi}\right)d\theta$$

$$= \int_0^{2\pi}\left(-a^2\overset{-1}{\cos\pi} - -a^2\overset{1}{\cos 0}\right)d\theta$$

$$= \int_0^{2\pi} 2a^2\,d\theta$$

$$= 2a^2\,\theta\,\Big|_0^{2\pi}$$

$$= 4\pi a^2 .$$

Def: assume that surface $\mathcal{S}$ is given by

$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$

for (u,v) in R: $a \le u \le b$, $l(u) \le v \le k(u)$.

assume function G is defined on $\mathcal{S}$. The surface integral of G over $\mathcal{S}$ is

$$\iint_{\mathcal{S}} G(P)\, dS = \iint_{R} G(P)\, |\vec{r}_u \times \vec{r}_v|\, dA .$$

Ex: Evaluate $\iint_{\mathcal{S}} z\, dS$, where $\mathcal{S}$ is the cylinder $x^2 + y^2 = 1$ cut by the planes $z = 0$ and $z = x + 1$.

Ex: Evaluate $\iint_{\mathcal{S}} y\, dS$, where $\mathcal{S}$ is that portion of the plane $x = 3$ cut by the cylinder $y^2 + (z-1)^2 = 1$ in the first octant.