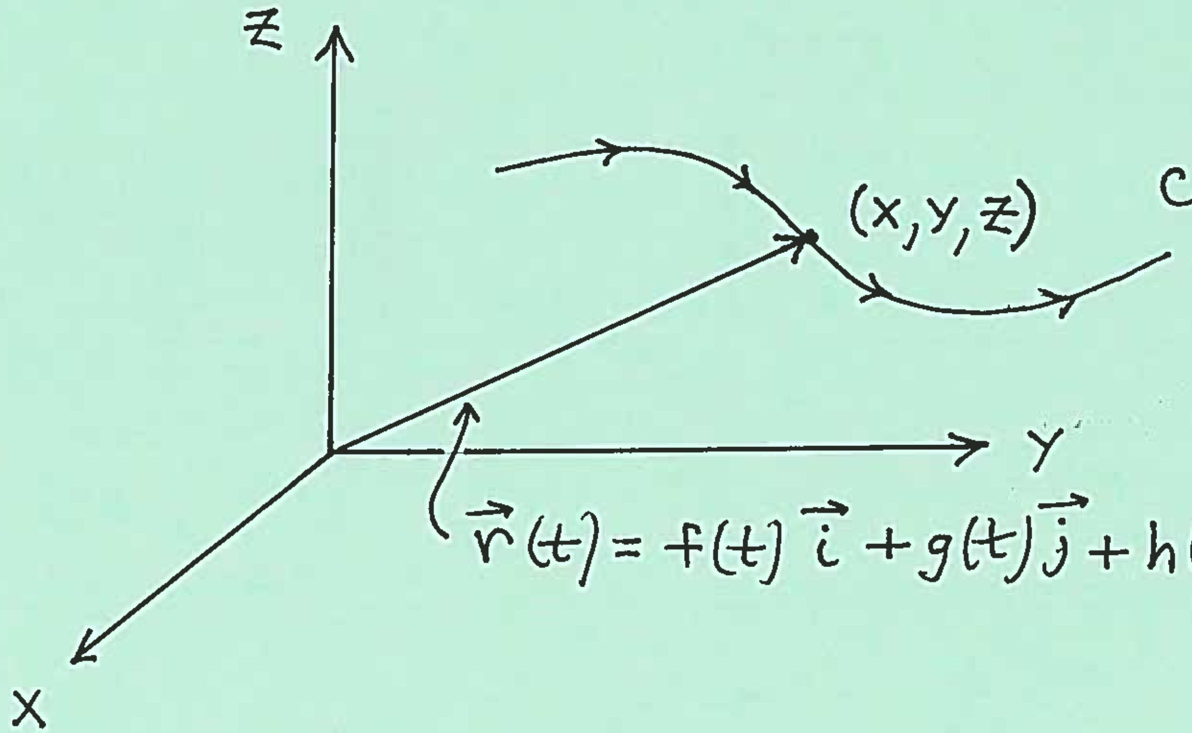
Section 16.6
Thomas Calculus
11th Ed.

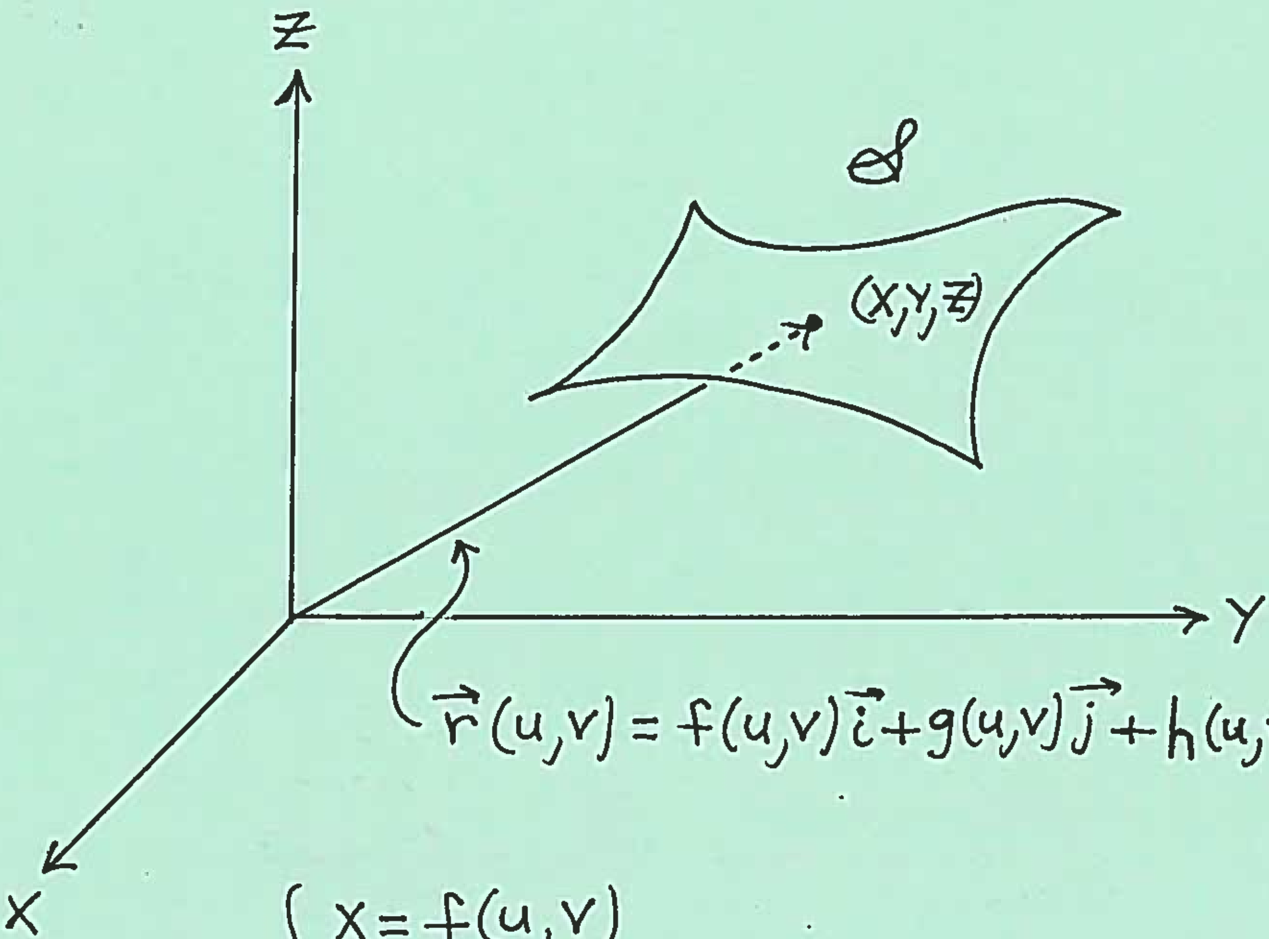
Parametrized Surfaces in 3D-Space

Recall: (Parametrized PATH C in 3D-Space)



$\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$

We want to generalize this parametrization to SURFACES in 3D-Space :



$$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$$

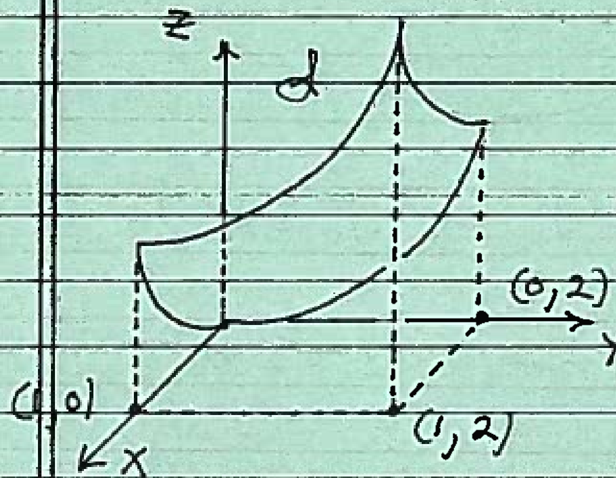
$$\mathcal{S}: \begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \end{cases}$$ for points

(u,v) in a region R in the xy-plane with

$$a \le u \le b, \quad \ell(u) \le v \le k(u).$$

Ex: Parametrize the following surfaces.

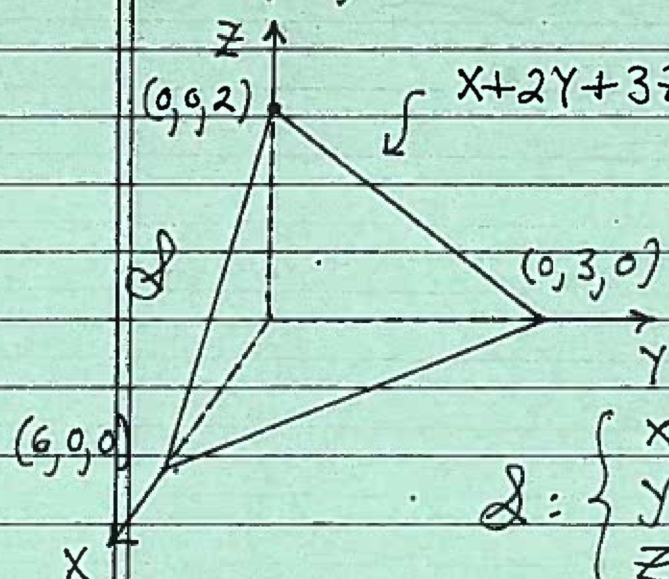
1.) S: portion of paraboloid $z = x^2 + y^2$ above rectangle with vertices $(0,0)$, $(1,0)$, $(0,2)$, and $(1,2)$:

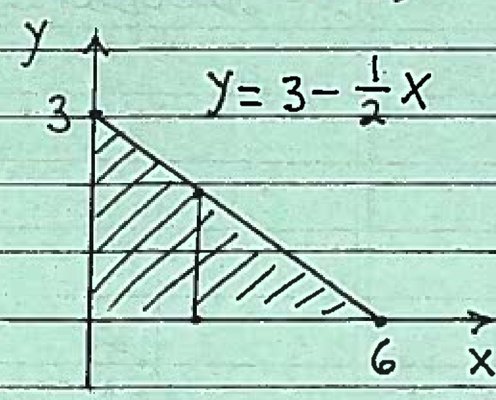


$$S: \begin{cases} x = u \\ y = v \\ z = u^2 + v^2 \end{cases}$$

for $0 \le u \le 1$, $0 \le v \le 2$

2.) S: triangle with vertices $(6,0,0)$, $(0,3,0)$, and $(0,0,2)$:

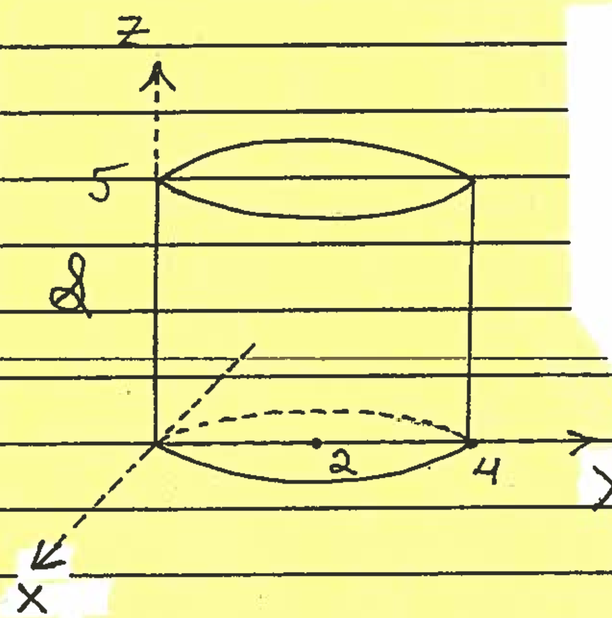


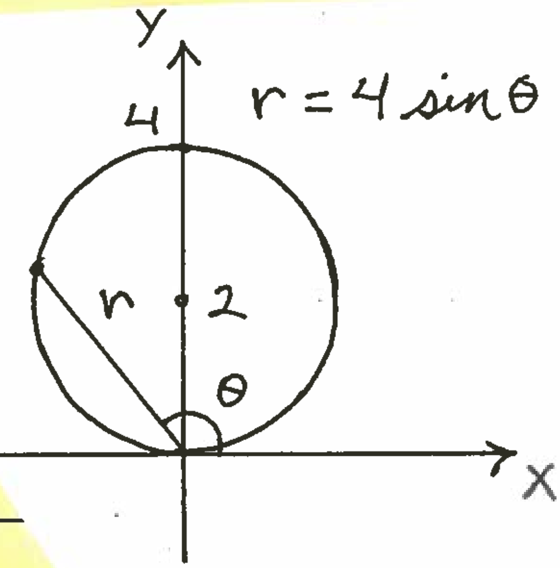


$$S: \begin{cases} x = u \\ y = v \\ z = \frac{1}{3}(6 - u - 2v) \end{cases}$$

for $0 \le u \le 6$, $0 \le v \le 3 - \frac{1}{2}u$

3.) S : surface of cylinder $x^2+(y-2)^2=4$ cut by the planes $z=0$ and $z=5$:





(Recall : (polar coordinates)

$$\left\{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right) \quad \text{then}$$

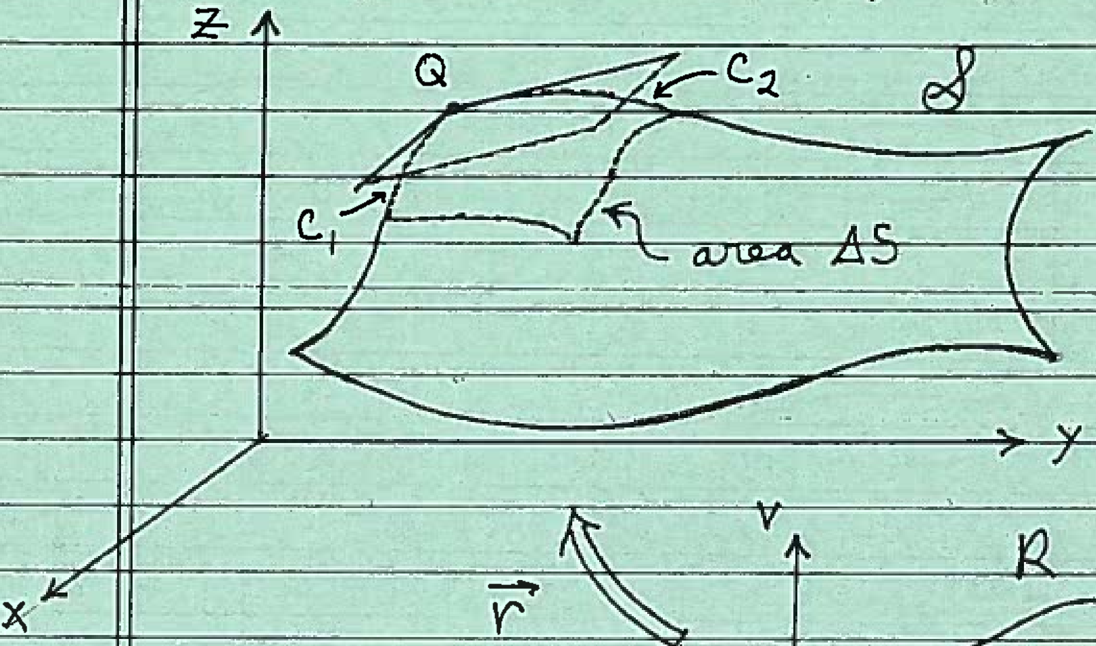
$$S: \begin{cases} x = r\cos\theta = (4\sin\theta)\cos\theta \\ y = r\sin\theta = (4\sin\theta)\sin\theta \quad \longrightarrow \\ z = t \end{cases}$$

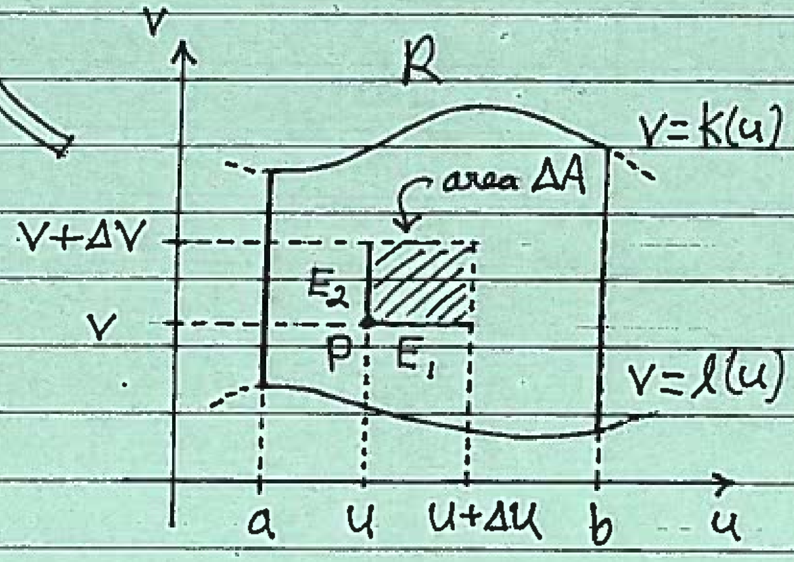
$$S: \begin{cases} x = 4\sin\theta\cos\theta \\ y = 4\sin^2\theta \\ z = t \end{cases}$$

for $0 \le \theta \le \pi$, $0 \le t \le 5$

Finding Area of Parametrized Surface

Let S be surface parametrized by
$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$
for (u,v) in R : $a \le u \le b$, $l(u) \le v \le k(u)$





assume that
$\vec{r}(P) = Q$;
let path C_1 be
the image of
edge E_1 under
$\vec{r}$ and let C_2
be the image of edge E_2 under $\vec{r}$;

(Recall : If $y=f(x)$, then $\Delta f \approx f'(x)\cdot\Delta x$.)

then

$$\text{path } C_1 \approx \Delta\vec{r} \text{ (along } E_1) \approx \vec{r}_u(Q)\cdot\Delta u$$

and

$$\text{path } C_2 \approx \Delta\vec{r} \text{ (along } E_2) \approx \vec{r}_v(Q)\cdot\Delta v \quad ;$$

Recall: area of ▱ formed by vectors $\vec{r}_u\cdot\Delta u$ and $\vec{r}_v\cdot\Delta v$ is (magnitude of cross product)

$$|\vec{r}_u\cdot\Delta u \times \vec{r}_v\cdot\Delta v| = |\vec{r}_u\times\vec{r}_v|\cdot\underbrace{\Delta v\cdot\Delta u}_{\Delta A} \quad ;$$

it follows that

$$\text{area } \Delta S \approx |\vec{r}_u\times\vec{r}_v|\cdot\Delta A \text{ , so that} \ldots$$

Def: If surface S is given by

$$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$$

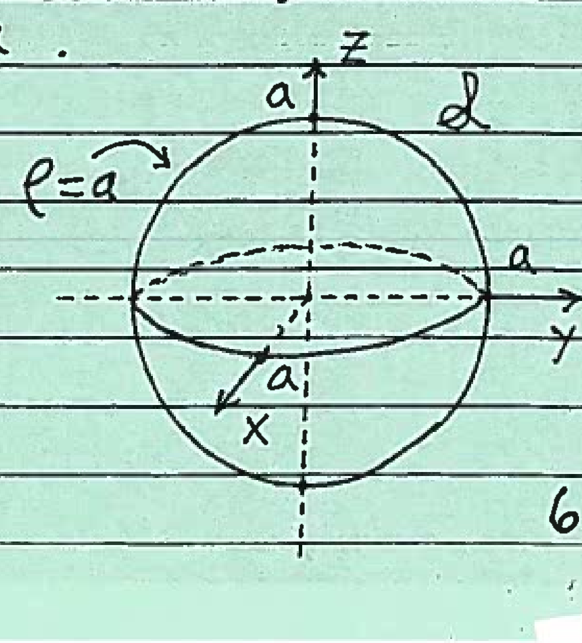
for (u,v) in R : $a \le u \le b$, $l(u) \le v \le k(u)$, then

$$\text{Area } S = \iint_S 1\, dS = \iint_R |\vec{r}_u\times\vec{r}_v|\, dA \quad ⊛$$

Ex: Find the surface area of a sphere of radius a.

(Recall: (spherical coordinates)

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases})$$



then

$$\mathcal{S}: \begin{cases} x = a\sin\phi\cos\theta \\ y = a\sin\phi\sin\theta \\ z = a\cos\phi \end{cases}$$

for $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$; now

$\mathcal{S}: \vec{r}(\phi, \theta)$ and

$$\begin{cases} \vec{r}_\theta = x_\theta \cdot \vec{i} + y_\theta \cdot \vec{j} + z_\theta \cdot \vec{k} \\ \quad = (-a\sin\phi\sin\theta)\vec{i} + (a\sin\phi\cos\theta)\vec{j} + (0)\vec{k}, \\ \vec{r}_\phi = x_\phi \cdot \vec{i} + y_\phi \cdot \vec{j} + z_\phi \cdot \vec{k} \\ \quad = (a\cos\phi\cos\theta)\vec{i} + (a\cos\phi\sin\theta)\vec{j} + (-a\sin\phi)\vec{k} : \end{cases}$$

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \end{vmatrix}$$

$$\begin{aligned} &= \left(-a^2\sin^2\phi\cos\theta - 0\right)\vec{i} \\ &\quad - \left(a^2\sin^2\phi\sin\theta - 0\right)\vec{j} \\ &\quad + \left(-a^2\sin\phi\cos\phi\sin^2\theta - a^2\sin\phi\cos\phi\cos^2\theta\right)\vec{k} \end{aligned}$$

$$\begin{aligned} &= \left(-a^2\sin^2\phi\cos\theta\right)\vec{i} \\ &\quad + \left(-a^2\sin^2\phi\sin\theta\right)\vec{j} \\ &\quad + \left(-a^2\sin\phi\cos\phi\,\underbrace{(\sin^2\theta + \cos^2\theta)}_{1}\right)\vec{k}, \end{aligned}$$

so that the magnitude of $|\vec{r}_\theta \times \vec{r}_\phi|$ is

$$|\vec{r}_\theta \times \vec{r}_\phi| = \sqrt{(-a^2\sin^2\phi\cos\theta)^2 + (-a^2\sin^2\phi\sin\theta)^2 + (-a^2\sin\phi\cos\phi)^2}$$

$$= \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{a^4\sin^4\phi\,\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + a^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{a^4\sin^2\phi\,(\underbrace{\sin^2\phi + \cos^2\phi}_{1})}$$

$$= a^2\sin\phi \quad ; \text{ then}$$

$$\text{Area } \mathcal{S} = \iint_{\mathcal{S}} 1\, dS = \iint_R |\vec{r}_\theta \times \vec{r}_\phi|\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \left(-a^2\cos\phi \,\Big|_{\phi=0}^{\phi=\pi}\right) d\theta$$

$$= \int_0^{2\pi} \left(-a^2\overset{-1}{\cancel{\cos\pi}} - -a^2\overset{1}{\cancel{\cos 0}}\right) d\theta$$

$$= \int_0^{2\pi} 2a^2\, d\theta$$

$$= 2a^2\,\theta\,\Big|_0^{2\pi}$$

$$= 4\pi a^2 .$$

Definition: assume that surface S is given by

$$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$$

for (u,v) in R: $a \le u \le b$, $l(u) \le v \le k(u)$. assume that function G is defined on S. The Surface Integral of G over S is

$$\boxed{\iint_S G(P)\,dS = \iint_R G(P)\,|\vec{r}_u \times \vec{r}_v|\,dA} \quad (*)$$

Example: $\iint_S z\,dS$, where S is the cylinder $x^2+y^2=1$ cut by the planes $z=0$ and $z=x+1$.

Let's first parametrize the surface S.

circle : $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$ for $0 \le \theta \le 2\pi$

and $z = t$ for $0 \le t \le x+1 = \cos\theta + 1$,

so

$$\boxed{\vec{r}(\theta, t) = (\cos\theta)\vec{i} + (\sin\theta)\vec{j} + (t)\vec{k}}\ .$$

Then

$$\vec{r}_\theta = (-\sin\theta)\vec{i} + (\cos\theta)\vec{j} + (0)\vec{k} \quad \text{and}$$

$$\vec{r}_t = (0)\vec{i} + (0)\vec{j} + (1)\vec{k} \longrightarrow$$

$$\vec{r}_\theta \times \vec{r}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\cos\theta)\vec{i} - (-\sin\theta)\vec{j} + (0)\vec{k}$$

$$= (\cos\theta)\vec{i} + (\sin\theta)\vec{j} \longrightarrow$$

$$|\vec{r}_\theta \times \vec{r}_t| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2}$$

$$= \sqrt{1} = 1\ ; \quad \text{now}$$

$$\iint_{S} z\, dS = \iint_{R} z \left| \vec{r}_\theta \times \vec{r}_t \right| dA$$

$$= \int_0^{2\pi} \int_0^{\cos\theta + 1} t \; (1) \; dt \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} t^2 \Big|_{t=0}^{t=\cos\theta+1} \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} (\cos\theta + 1)^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\cos^2\theta + 2\cos\theta + 1 \right) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\frac{1}{2}(1 + \cos 2\theta) + 2\cos\theta + 1 \right) d\theta$$

$$= \frac{1}{2} \left[\frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + 2\sin\theta + \theta \right] \Big|_0^{2\pi}$$

$$= \frac{1}{2} \left[\frac{3}{2}\theta + \frac{1}{4}\sin 2\theta + 2\sin\theta \right] \Big|_0^{2\pi}$$

$$= \frac{1}{2} \left[\frac{3}{2}(2\pi) + \frac{1}{4}\overset{0}{\cancel{\sin}(4\pi)} + 2\overset{0}{\cancel{\sin}(2\pi)} \right]$$

$$= \frac{3}{2}\pi$$