Section 13.4
Thomas Calculus
11th Ed.

## The Principal Unit Normal Vector to Path C in Space

Consider a Vector Function $\vec{r}(t)$ and its Unit Tangent Vector

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}.$$

The <u>Principal</u> <u>Unit</u> <u>Normal</u> <u>Vector</u> to path C is

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{\frac{d\vec{T}(t)}{dt}}{\left|\frac{d\vec{T}(t)}{dt}\right|}.$$

FACTS : 1.) $\vec{N}(t)$ is a unit vector.

2.) $\vec{N}(t)$ is orthogonal to the Unit Tangent Vector $\vec{T}(t)$.

3.) $\vec{N}(t)$ points in the direction that path C is turning.

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1.) Obvious.

2.) $\vec{T}(t)$ is a unit vector, so

$$|\vec{T}(t)| = 1 \rightarrow |\vec{T}(t)|^2 = 1^2 \rightarrow$$

$$\vec{T}(t) \cdot \vec{T}(t) = 1 \xrightarrow{D}$$

$$\vec{T}(t) \cdot \vec{T}'(t) + \vec{T}'(t) \cdot \vec{T}(t) = 0 \rightarrow$$

$$2\left(\vec{T}'(t) \cdot \vec{T}(t)\right) = 0 \rightarrow$$

$$\vec{T}'(t) \cdot \vec{T}(t) = 0 \rightarrow$$

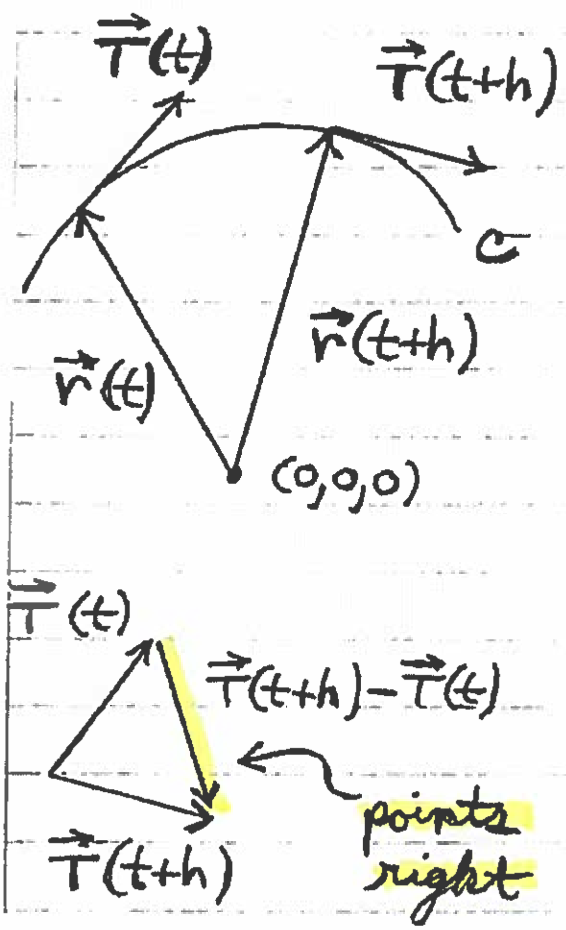
$$\frac{\vec{T}'(t)}{|\vec{T}'(t)|} \cdot \vec{T}(t) = 0 \rightarrow$$

$$\vec{N}(t) \cdot \vec{T}(t) = 0,$$

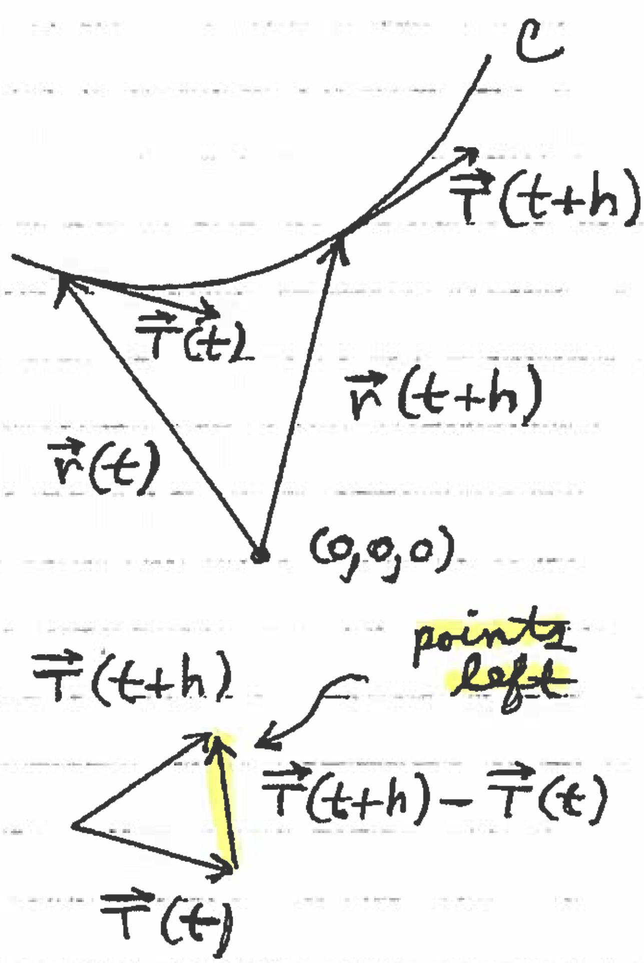
i.e., $\vec{N}(t)$ is orthogonal to $\vec{T}(t)$.

3.) $$\vec{T}\,'(t) = \lim_{h \to 0} \frac{\vec{T}(t+h) - \vec{T}(t)}{h} :$$

(turning right)




(turning left)



Example: Consider the vector function $\vec{r}(t) = (t^2)\vec{i} + (t^3)\vec{j}$ for $t \geq 0$.

1.) Plot path C traced out by $\vec{r}(t)$ in 2D-Space.

2.) Find $\vec{v}(t)$, $\vec{a}(t)$, $\vec{T}(t)$, and $\vec{N}(t)$.

3.) Sketch path C and plot $\vec{r}(1)$, $\vec{v}(1)$, $\vec{a}(1)$, $\vec{T}(1)$, and $\vec{N}(1)$. also find the speed and acceleration of motion along path C when $t=1$.

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$$\begin{cases} x = t^2 \\ y = t^3 \rightarrow t = y^{1/3} \end{cases} \longrightarrow x = (y^{1/3})^2 = y^{2/3}$$

or $\quad y = x^{3/2}$.

1.) Later.

2.) $\vec{r}(t) = (t^2)\vec{i} + (t^3)\vec{j} \xrightarrow{D}$

$\vec{v}(t) = (2t)\vec{i} + (3t^2)\vec{j} \xrightarrow{D}$

$\vec{a}(t) = (2)\vec{i} + (6t)\vec{j}$ ;

$$|\vec{v}(t)| = \sqrt{(2t)^2 + (3t^2)^2}$$

$$= \sqrt{4t^2 + 9t^4}$$

$$= \sqrt{t^2(4+9t^2)} \longrightarrow$$

$$\boxed{|\vec{v}(t)| = t\sqrt{4+9t^2}}\text{, so}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

$$= \frac{2t}{t\sqrt{4+9t^2}}\vec{i} + \frac{3t^2}{t\sqrt{4+9t^2}}\vec{j}\text{, i.e.,}$$

$$\boxed{\vec{T}(t) = \frac{2}{\sqrt{4+9t^2}}\vec{i} + \frac{3t}{\sqrt{4+9t^2}}\vec{j}}\ ;$$

$$\vec{T}(t) = 2(4+9t^2)^{-1/2}\,\vec{i} + \frac{3t}{(4+9t^2)^{1/2}}\,\vec{j} \xrightarrow{D}$$

$$\vec{T}\,'(t) = \cancel{2}\cdot\frac{-1}{\cancel{2}}(4+9t^2)^{-3/2}\cdot(18t)\,\vec{i}$$

$$+\ \frac{(4+9t^2)^{1/2}(3) - (3t)\cdot\frac{1}{\cancel{2}}(4+9t^2)^{-1/2}\cdot(\cancel{18}^{\,9}t)}{\left[(4+9t^2)^{1/2}\right]^2}\,\vec{j}$$

$$= \frac{-18t}{(4+9t^2)^{3/2}}\,\vec{i} + \frac{\dfrac{3(4+9t^2)^{1/2}}{1} - \dfrac{27t^2}{(4+9t^2)^{1/2}}}{4+9t^2}\,\vec{j}$$

$$= \frac{-18t}{(4+9t^2)^{3/2}}\,\vec{i} + \left[\frac{3(4+9t^2) - 27t^2}{(4+9t^2)^{1/2}}\cdot\frac{1}{(4+9t^2)}\right]\vec{j}$$

$$= \frac{-18t}{(4+9t^2)^{3/2}}\,\vec{i} + \frac{12 + \cancel{27t^2} - \cancel{27t^2}}{(4+9t^2)^{3/2}}\,\vec{j} \longrightarrow$$

$$\vec{T}\,'(t) = \frac{-18t}{(4+9t^2)^{3/2}}\,\vec{i} + \frac{12}{(4+9t^2)^{3/2}}\,\vec{j},$$ *then*

$$|\vec{T}\,'(t)| = \sqrt{\left(\frac{-18t}{(4+9t^2)^{3/2}}\right)^2 + \left(\frac{12}{(4+9t^2)^{3/2}}\right)^2}$$

$$= \sqrt{\frac{324\,t^2}{(4+9t^2)^3} + \frac{144}{(4+9t^2)^3}}$$

$$= \sqrt{\frac{36\cancel{(4+9t^2)}}{(4+9t^2)^{\cancel{3}\,2}}} = \frac{6}{4+9t^2} \text{ ; now}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{\frac{-18t}{(4+9t^2)^{3/2}}}{\frac{6}{4+9t^2}}\vec{i} + \frac{\frac{12}{(4+9t^2)^{3/2}}}{\frac{6}{4+9t^2}}\vec{j}$$

$$= \frac{-18t}{(4+9t^2)^{3/2}} \cdot \frac{4+9t^2}{6}\vec{i} + \frac{12}{(4+9t^2)^{3/2}} \cdot \frac{4+9t^2}{6}\vec{j}$$

$$= \frac{-3t}{\sqrt{4+9t^2}}\vec{i} + \frac{2}{\sqrt{4+9t^2}}\vec{j}, \text{ i.e.,}$$

$$\boxed{\vec{N}(t) = \frac{-3t}{\sqrt{4+9t^2}}\vec{i} + \frac{2}{\sqrt{4+9t^2}}\vec{j}}$$

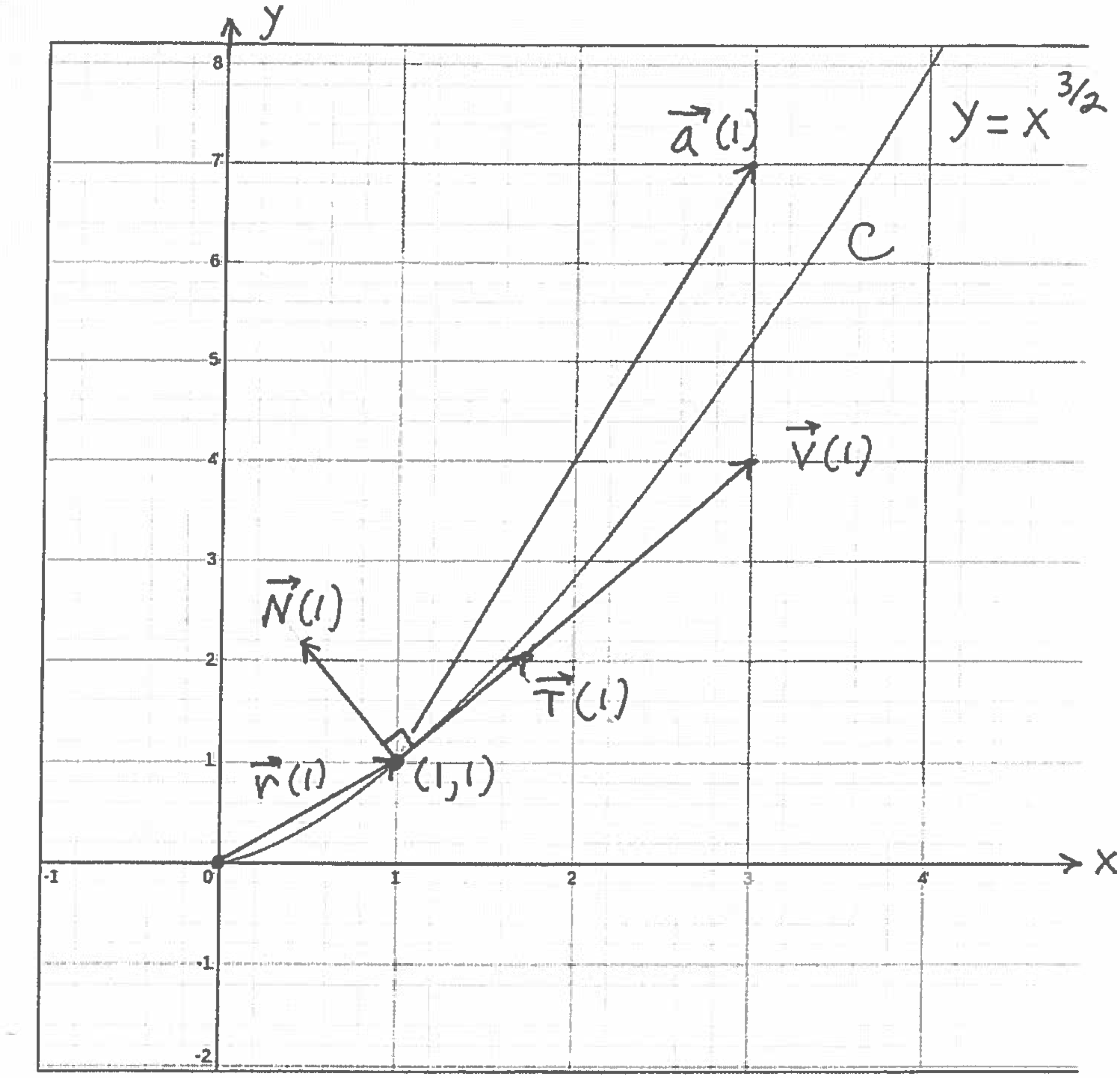
3.) $$\vec{r}(1) = (1)\vec{i} + (1)\vec{j} ,$$
$$\vec{v}(1) = 2\vec{i} + 3\vec{j} ,$$
$$\vec{a}(1) = 2\vec{i} + 6\vec{j} ,$$
$$\vec{T}(1) = \frac{2}{\sqrt{13}}\vec{i} + \frac{3}{\sqrt{13}}\vec{j} \approx 0.55\vec{i} + 0.83\vec{j} ,$$
$$\vec{N}(1) = \frac{-3}{\sqrt{13}}\vec{i} + \frac{2}{\sqrt{13}}\vec{j} \approx -0.83\vec{i} + 0.55\vec{j} ;$$

speed of motion is

$$\frac{ds}{dt} = |\vec{v}(1)| = \sqrt{2^2+3^2} = \sqrt{13} ;$$

acceleration of motion is

$$a(1) = \frac{d^2s}{dt^2} = \vec{T}(1) \cdot \vec{a}(1)$$
$$= \left(\frac{2}{\sqrt{13}}\vec{i} + \frac{3}{\sqrt{13}}\vec{j}\right)\left(2\vec{i} + 6\vec{j}\right)$$
$$= \frac{4}{\sqrt{13}} + \frac{18}{\sqrt{13}} = \frac{22}{\sqrt{13}} .$$


y
x
$y = x^{3/2}$
C
$\vec{a}(1)$
$\vec{v}(1)$
$\vec{N}(1)$
$\vec{T}(1)$
$\vec{r}(1)$
(1,1)
8
7
6
5
4
3
2
1
-1
-2
-1
0
1
2
3
4