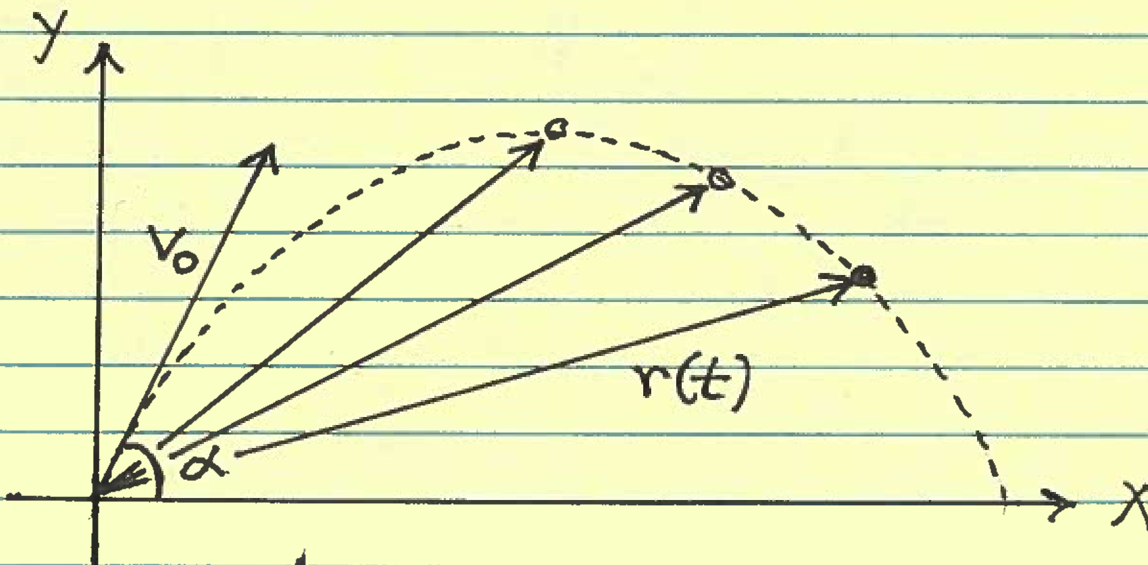
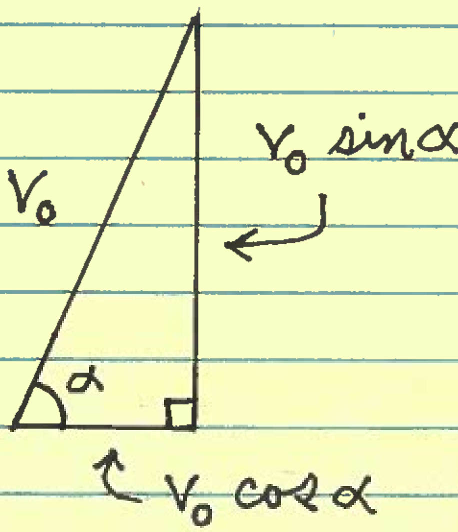
Section 13.2
Thomas Calculus
11th Ed.

Vectors and Projectile Motion

Consider an object projected from the origin with an initial velocity v_0 and at an angle α. We want to model the object's path through space using a vector function.





From Trigonometry we have the vertical and horizontal components of the initial velocity.

Assuming that gravity is the only force acting on the object, then the

HORIZONTAL COMPONENT of motion at time t is:

$$(\text{distance}) = \begin{pmatrix}\text{constant}\\ \text{velocity}\end{pmatrix}(\text{time})$$

$$\rightarrow \quad \boxed{x(t) = (v_0 \cos\alpha)\, t} \ ;$$

and the

VERTICAL COMPONENT of motion at time t is:

(We will assume that the acceleration due to gravity at sea level is

$$g = 32 \, \frac{\text{ft.}}{\text{sec}^2} \quad \text{OR} \quad g = 9.8 \, \frac{\text{m}}{\text{sec}^2} \ )$$

$$(\text{height}) = -\frac{1}{2} g t^2 + \begin{pmatrix}\text{initial}\\ \text{velocity}\end{pmatrix} t + (\text{initial height}) \rightarrow$$

$$\boxed{y(t) = -\frac{1}{2} g t^2 + (v_0 \sin\alpha)\, t + y(0)} \ .$$

Thus, the position vector for the object's path of motion is

$$\vec{r}(t) = (v_0(\cos\alpha)t)\,\vec{i}$$
$$+ \left(-\frac{1}{2}gt^2 + v_0(\sin\alpha)t + y(0)\right)\vec{j}\ .$$

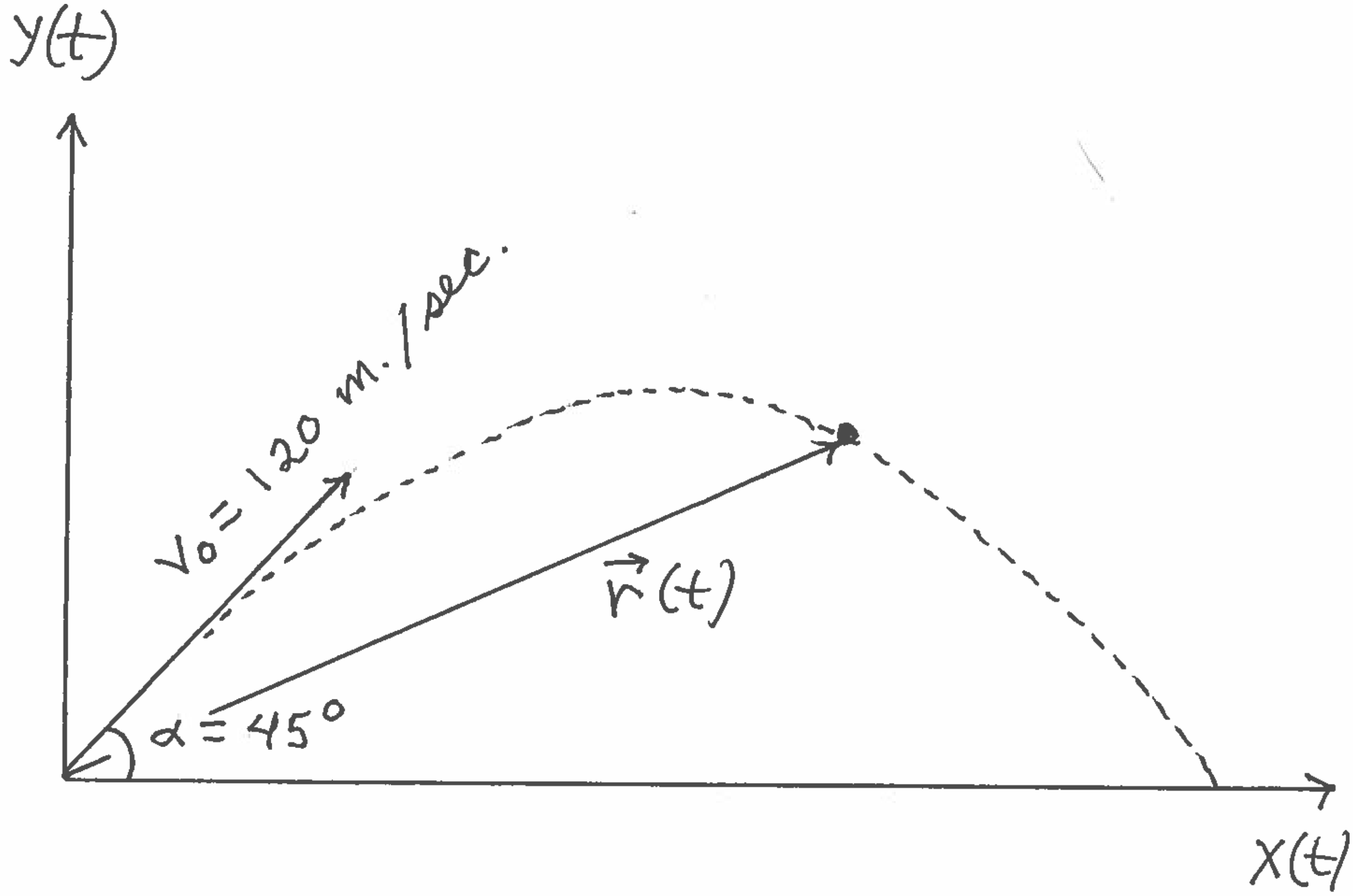
Example: a bullet is fired from a muzzle loader rifle at ground level with initial velocity 120 m./sec and at an angle of 45°.

1.) How long is the bullet in the air until it strikes the ground?

2.) How far away will the bullet strike the ground?

3.) How high does the bullet go?

4.) What is the bullet's speed at its highest point?



The bullet's position vector is

$$\vec{r}(t) = x(t)\cdot \vec{i} + y(t)\,\vec{j}$$

$$= \left(120(\cos 45°)t\right)\vec{i}$$

$$+ \left(-\tfrac{1}{2}(9.8)t^2 + 120(\sin 45°)t + (0)\right)\vec{j}$$

$$= \left(120\left(\sqrt{2}/2\right)t\right)\vec{i}$$

$$+ \left(-4.9t^2 + 120\left(\tfrac{\sqrt{2}}{2}\right)t\right)\vec{j}$$

$= (60\sqrt{2})t \cdot \vec{i} + (-4.9t^2 + (60\sqrt{2})t)\vec{j}$, i.e.,

$$\boxed{\vec{r}(t) = (60\sqrt{2})t \cdot \vec{i} + (-4.9t^2 + (60\sqrt{2})t)\vec{j}} \; ;$$

the bullet's <u>velocity</u> <u>vector</u> is

$$\boxed{\vec{v}(t) = \vec{r}\,'(t) = (60\sqrt{2})\vec{i} + (-9.8t + 60\sqrt{2})\vec{j}} \; .$$

1.) Bullet Strikes Ground :

Set $y(t) = 0 \rightarrow -4.9t^2 + (60\sqrt{2})t = 0$

$\rightarrow t(-4.9t + 60\sqrt{2}) = 0$

$\rightarrow t = 0$ or $-4.9t + 60\sqrt{2} = 0$

$\rightarrow 4.9t = 60\sqrt{2}$

$\rightarrow t = \dfrac{60\sqrt{2}}{4.9} \approx 17.32$ sec.

2.) How Far away Strike Ground :

$x\left(\dfrac{60\sqrt{2}}{4.9}\right) = 60\sqrt{2}\left(\dfrac{60\sqrt{2}}{4.9}\right) = \dfrac{7200}{4.9} \approx 1469$ m.

3.) How High: Set $y'(t) = 0 \rightarrow$

$$-9.8t + 60\sqrt{2} = 0 \rightarrow$$

$$9.8t = 60\sqrt{2} \rightarrow$$

$$t = \frac{60\sqrt{2}}{9.8} \approx 8.66 \text{ sec.}$$; then

maximum height is

$$y\left(\frac{60\sqrt{2}}{9.8}\right) = -4.9\left(\frac{60\sqrt{2}}{9.8}\right)^2 + 60\sqrt{2}\left(\frac{60\sqrt{2}}{9.8}\right)$$

$$\approx 367.3 \text{ m.}$$

4.) Speed at Highest Point:

Velocity Vector at highest point is

$$\vec{V}\left(\frac{60\sqrt{2}}{9.8}\right) = (60\sqrt{2})\vec{i}$$
$$+ \left(\cancel{-9.8}\left(\frac{60\sqrt{2}}{\cancel{9.8}}\right) + 60\sqrt{2}\right)\vec{j}$$
$$= (60\sqrt{2})\vec{i} + (0)\vec{j}$$, so

speed at highest point is

$$\left|\vec{V}\left(\frac{60\sqrt{2}}{9.8}\right)\right| = 60\sqrt{2} \approx 84.85 \text{ m./sec.}$$

<u>Example</u>: A golf ball is hit from the top of Kerr Hall (64 ft. high) with an initial velocity of 96 ft./sec. at an angle of 30°.

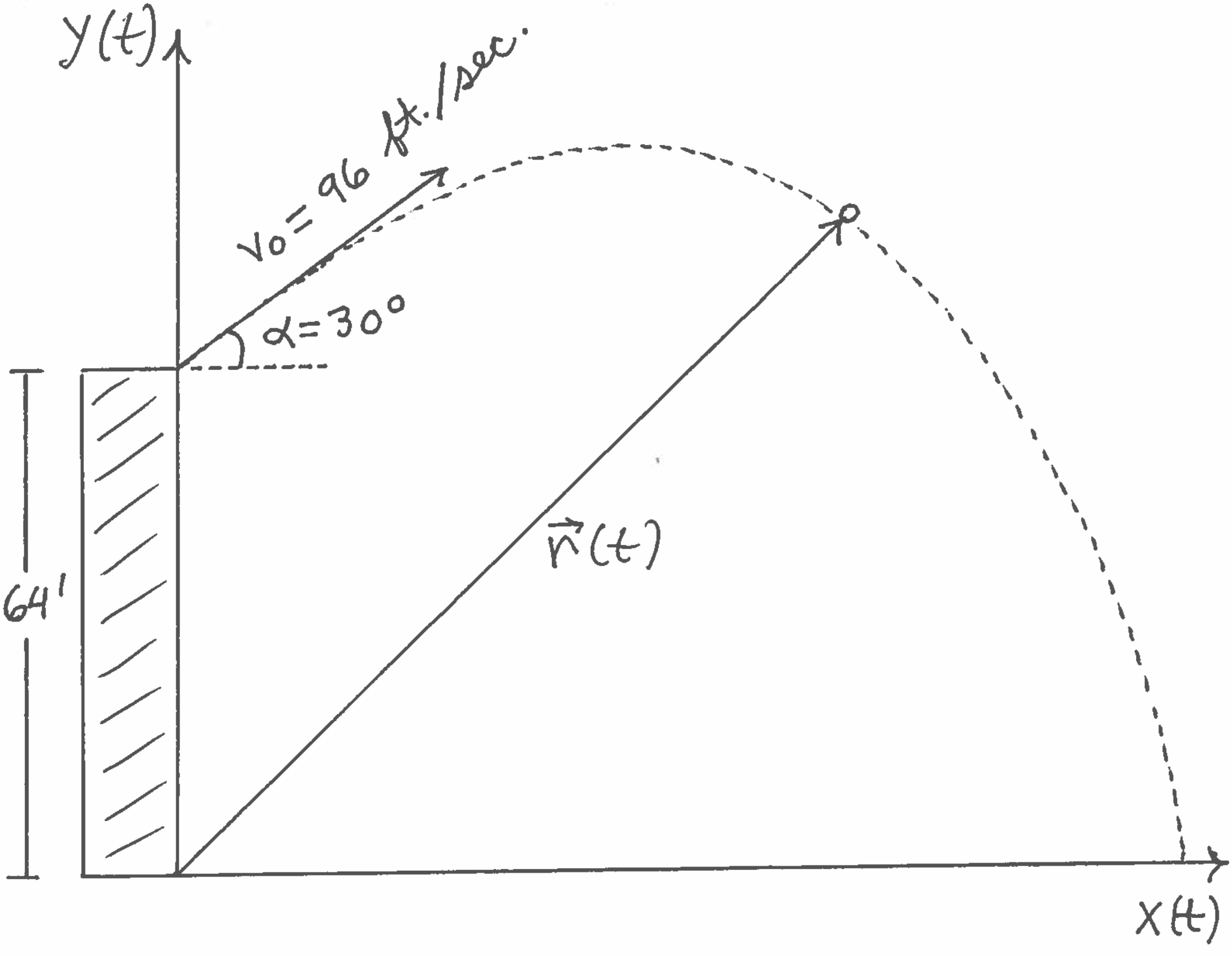
1.) What is the position and speed of the ball when $t=2$ sec.?

2.) How high does the ball go?

3.) How long is the ball in the air before it strikes the ground?

4.) How far from the base of the building will the ball strike the ground?

5.) What is the ball's direction of motion (velocity vector), speed of motion, and angle formed as it strikes the ground?



The ball's <u>position</u> <u>vector</u> is

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$$

$$= (96(\cos 30°)t)\,\vec{i}$$

$$+ \left(-\tfrac{1}{2}(32)t^2 + 96(\sin 30°)t + 64\right)\vec{j}$$

$$= \left(96\left(\tfrac{\sqrt{3}}{2}\right)t\right)\vec{i}$$

$$+ \left(-16t^2 + 96\left(\tfrac{1}{2}\right)t + 64\right)\vec{j}$$

$$= (48\sqrt{3})t\cdot\vec{i} + (-16t^2 + 48t + 64)\,\vec{j}, \text{ i.e.,}$$

$$\boxed{\vec{r}(t) = (48\sqrt{3})t\cdot\vec{i} + (-16t^2 + 48t + 64)\,\vec{j}} \;;$$

the ball's velocity vector is

$$\boxed{\vec{v}(t) = \vec{r}\,'(t) = (48\sqrt{3})\,\vec{i} + (-32t + 48)\,\vec{j}} \;.$$

1.) If $t = 2$ sec., then the position is $\vec{r}(2) = (48\sqrt{3})(2)\,\vec{i}$

$+ (-16(2)^2 + 48(2) + 64)\,\vec{j}$

$= (96\sqrt{3})\,\vec{i} + (96)\,\vec{j}$

$\approx (166.3)\,\vec{i} + (96)\,\vec{j}$, so

$x \approx 166.3$ ft., $y = 96$ ft. ;

the velocity vector is

$\vec{v}(2) = (48\sqrt{3})\,\vec{i} + (-32(2) + 48)\,\vec{j}$

$= (48\sqrt{3})\,\vec{i} + (-16)\,\vec{j}$, so

speed is

$|\vec{v}(2)| = \sqrt{(48\sqrt{3})^2 + (-16)^2}$

$= \sqrt{6912 + 256} = \sqrt{7168}$

≈ 84.7 ft./sec.

2.) How high does ball go?:

Set $y'(t) = 0 \rightarrow -32t + 48 = 0$

$\rightarrow 32t = 48 \rightarrow t = \frac{48}{32} = \frac{3}{2}$ sec.;

then maximum height is

$$Y\left(\frac{1}{2}\right) = -16\left(\frac{1}{2}\right)^2 + 48\left(\frac{1}{2}\right) + 64$$

$$= -4 + 24 + 64$$

$$= 84 \text{ ft.}$$

3.) Ball Strikes Ground:

Set $y(t) = 0 \rightarrow -16t^2 + 48t + 64 = 0$

$\rightarrow -16(t^2 - 3t - 4) = 0 \rightarrow$

$-16(t-4)(t+1) = 0 \rightarrow t = 4$ sec.

4.) How Far From Building?

$X(4) = 48\sqrt{3}(4) \approx 332.55$ ft.

5.) Direction of motion when $t = 4$ sec. is

a.) $\vec{v}(4) = (48\sqrt{3})\vec{i} + (-32(4) + 48)\vec{j}$

$= (48\sqrt{3})\vec{i} + (-80)\vec{j}$, so the

b.) speed of motion is

$$|\vec{v}(4)| = \sqrt{(48\sqrt{3})^2 + (-80)^2}$$
$$= \sqrt{6912 + 6400}$$
$$= \sqrt{13,312} \approx 115.4 \text{ ft./sec.}$$

c.) Ball strikes ground at angle

$\theta = \arctan\left(\frac{80}{48\sqrt{3}}\right)$

$\approx 43.9°$