Math 21D
Kouba
Stoke's Theorem

Recall: (2D-Space) If $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ is a velocity field defined on region R, then the k-component of curl of $\vec{F}$ at point (x,y) in R is

$$\text{k-comp. of curl of } \vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

Def: Let $\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$ be a vector field defined on surface S. The curl of $\vec{F}$ is the vector field

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}.$$

Remark: It can be shown that the magnitude of the curl of $\vec{F}$ at the point (x,y,z) on surface S is a measure of the rate of rotation of the fluid relative to a vector $\perp$ to the tangent plane at point (x,y,z).

Ex: Find the curl of
$\vec{F}(x,y,z) = (x^2+y^2)\vec{i} + (y^2+z^2)\vec{j} + (x^2+z^2)\vec{k}$:

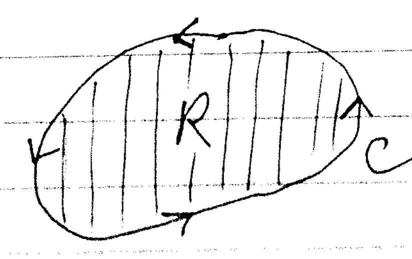
$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & y^2+z^2 & x^2+z^2 \end{vmatrix}$$

$$= (0-2z)\vec{i} - (2x-0)\vec{j} + (0-2y)\vec{k}$$
$$= (-2z)\vec{i} + (-2x)\vec{j} + (-2y)\vec{k}.$$

Circulation on Loop C in 3D-Space

Recall: (Green's Theorem – Circulation Curl Form – in 2D-Space) Assume that $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ is defined on region R enclosed by loop C in 2D-Space. Then the circulation of $\vec{F}$ around C is

$$\text{Circ} = \oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$



Stoke's Theorem will generalize this concept in 3D-Space.

Recall: If $G(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ is defined on surface S, then the flux of $\vec{G}$ across S is

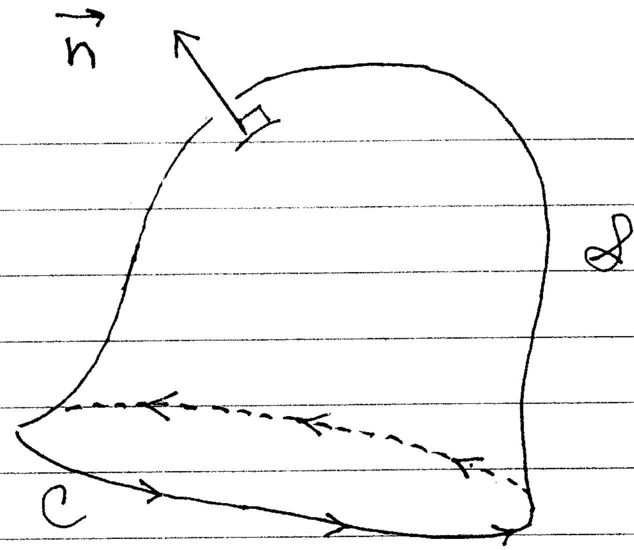
$$\text{Flux} = \iint_S \vec{G} \cdot \vec{n}\, dS,$$

where $\vec{n}$ is a chosen unit normal vector to surface S.

Stoke's Theorem: Let

$\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ be a vector field defined on surface S with boundary C oriented counter-clockwise relative to the surface's unit normal vector $\vec{n}$ (right-hand rule). Then the circulation of $\vec{F}$ around C is

$$\begin{aligned} \text{Circ} &= \oint_C \vec{F} \cdot \vec{T}\, ds \\ &= \underbrace{\iint_S \underbrace{\vec{\nabla} \times \vec{F}}_{\text{curl } \vec{F}} \cdot \vec{n}\, dS}_{\text{Flux of the curl of } \vec{F}} \end{aligned}$$



Ex : Verify Stoke's Theorem for

$\vec{F}(x,y,z) = (xy^2)\vec{i} + (y)\vec{j} + (yz)\vec{k}$

and

S : top half of sphere $x^2+y^2+z^2=1$