Section 16.7
Thomas Calculus
11th Ed.

Stoke's Theorem

Recall: (In 2D-Space)

If $F(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ is a velocity vector field defined on a region R, then the k-component of curl (also called circulation density) at point (x,y) in R is

$$(\text{curl}\,\vec{F})\cdot\vec{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Definition: Let

$$\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$$

be a vector field defined on surface S. The curl of $\vec{F}$ is the vector field

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

Remark: It can be shown that the magnitude of the curl of $\vec{F}$ at point (x, y, z) on the surface of $\mathcal{S}$ is a measure of the rate of rotation of fluid relative to a vector perpendicular to the tangent plane at point (x, y, z).

Example: Find the curl of

$$\vec{F}(x, y, z) = (x^2 + y^2)\vec{i} + (y^2 + z^2)\vec{j} + (x^2 + z^2)\vec{k}.$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & y^2+z^2 & x^2+z^2 \end{vmatrix}$$

$$= (0-2z)\vec{i} - (2x-0)\vec{j} + (0-2y)\vec{k}$$

$$= (-2z)\vec{i} + (-2x)\vec{j} + (-2y)\vec{k}$$

Circulation on a Loop in 3D-Space

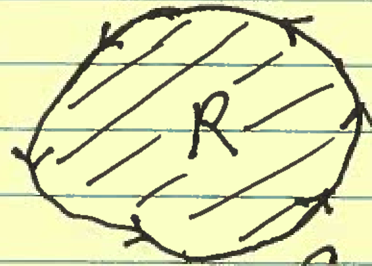
Recall: Green's Theorem 2 (circulation-curl form in 2D-Space) —

assume that

$$\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$$

is a vector field defined on region R enclosed by loop C in 2D-Space. Then the circulation (flow) of $\vec{F}$ around loop C is

$$\text{Flow} = \oint_C \vec{F} \cdot \vec{T}\, ds$$



$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Stoke's Theorem generalizes this concept in 3D-Space.

<u>Recall</u>: If

$$G(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$$

is defined on surface S in 3D-Space, then the flux of $\vec{G}$ across S is

$$\text{Flux} = \iint_S \vec{G} \cdot \vec{n}\, dS ,$$

where $\vec{n}$ is a chosen unit normal vector to $\mathcal{S}$.

<u>Stoke's Theorem</u>: Let

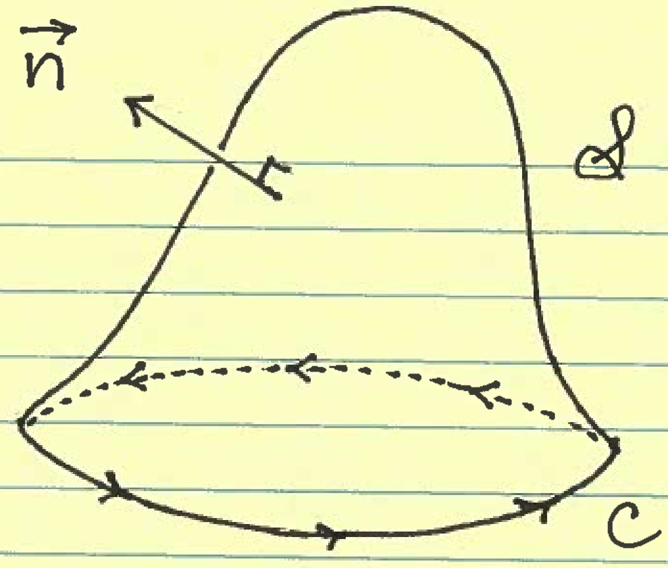
$$\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$$

be a vector field defined on a surface $\mathcal{S}$ in 3D-Space with a closed boundary C (oriented counter-clockwise relative to the surface's unit normal vector $\vec{n}$).

Then the circulation (<u>flow</u>) of $\vec{F}$ around loop C is

$$\text{Flow} = \oint_C \vec{F} \cdot \vec{T}\, ds$$

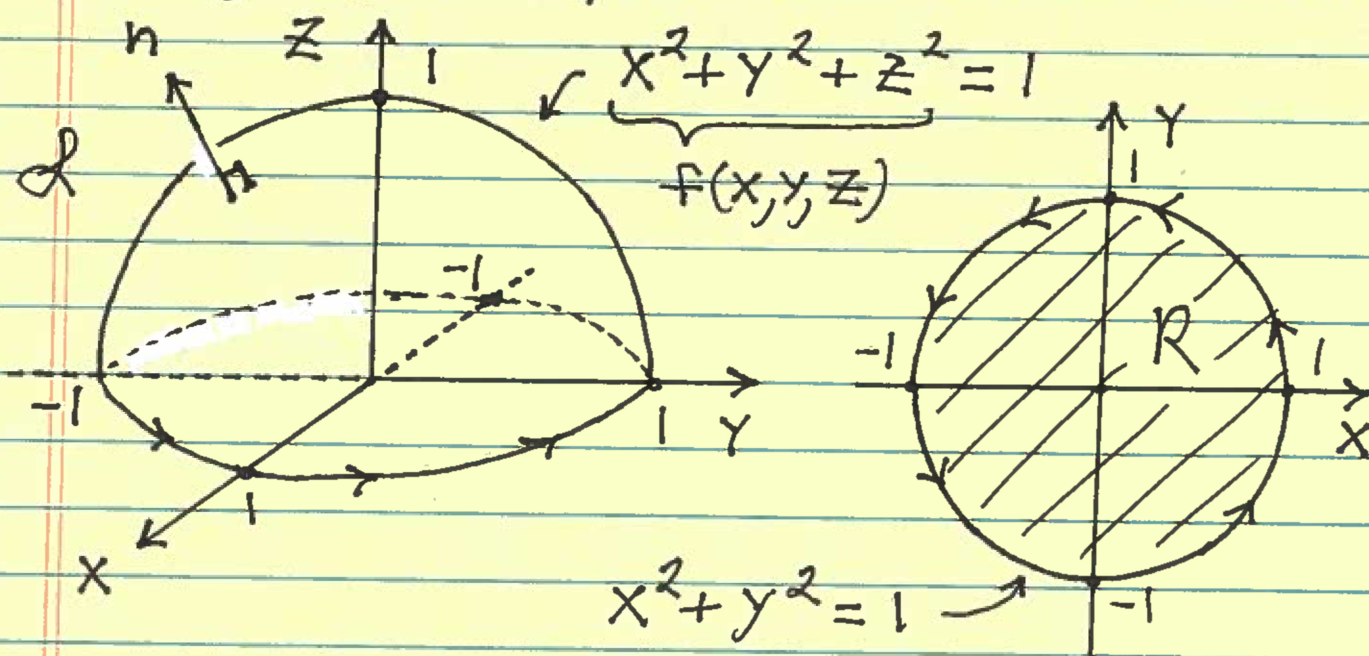
$$= \iint_{\mathcal{S}} \underbrace{\vec{\nabla} \times \vec{F}}_{\text{curl } \vec{F}} \cdot \vec{n}\, dS .$$



<u>Example</u>: Verify Stoke's Theorem for

$$\vec{F}(x,y,z) = (xy^2)\,\vec{i} + (y)\,\vec{j} + (yz)\,\vec{k}$$

and surface $\mathcal{S}$: the top half of the sphere $x^2+y^2+z^2=1$.



$$C: \begin{cases} x = \cos t \\ y = \sin t \\ z = 0 \end{cases} \quad \text{for } 0 \le t \le 2\pi$$

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C M\, dx + N\, dy + P\, dz$$

$$= \oint_C \left[(xy^2)\frac{dx}{dt} + (y)\frac{dy}{dt} + (yz)\frac{dz}{dt} \right]$$

$$= \int_0^{2\pi} \left[(\cos t)(\sin t)^2(-\sin t) + (\sin t)(\cos t) + 0 \right] dt$$

$$= \int_0^{2\pi} \left[-\sin^3 t \cos t + \sin t \cdot \cos t \right] dt$$

$$= \left(-\frac{1}{4}\sin^4 t + \frac{1}{2}\sin^2 t \right) \Big|_0^{2\pi}$$

$$= \left(-\frac{1}{4}\underbrace{\sin^4(2\pi)}_{0} + \frac{1}{2}\underbrace{\sin^2(2\pi)}_{0} \right)$$

$$- \left(-\frac{1}{4}\underbrace{\sin^4(0)}_{0} + \frac{1}{2}\underbrace{\sin^2(0)}_{0} \right) = 0;$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & y & yz \end{vmatrix} = (z - 0)\,\vec{i}$$

$$- (0 - 0)\,\vec{j} + (0 - 2xy)\,\vec{k}$$

$= (z)\vec{i} + (-2xy)\vec{k}$; for surface:

$\vec{\nabla} f = (2x)\vec{i} + (2y)\vec{j} + (2z)\vec{k} \longrightarrow$

$|\vec{\nabla} f| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2}$

$= \sqrt{4(x^2 + y^2 + z^2)}$

$= \sqrt{4(1)} = 2$; normal

vector : $\vec{n} = \dfrac{\vec{\nabla} f}{|\nabla f|} = (x)\vec{i} + (y)\vec{j} + (z)\vec{k}$;

$$\iint_S \vec{\nabla} \times \vec{F} \cdot \vec{n}\, dS = \iint_S (xz + 0 - 2xyz)\, dS$$

$$= \iint_R (xz - 2xyz) \cdot \frac{|\vec{\nabla} f|}{|f_z|}\, dA$$

$$= \iint_R (xz - 2xyz) \cdot \frac{\cancel{2}}{\cancel{2z}}\, dA$$

$$= \iint_R (x - 2xy)\, dA$$

$$R: \begin{cases} x = \cos\theta \\ y = \sin\theta, \ 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \end{cases}$$

$$= \int_0^{2\pi}\int_0^1 [r\cos\theta - 2(r\cos\theta)(r\sin\theta)]\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 [r^2\cos\theta - 2r^3\sin\theta\cos\theta]\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\left(\tfrac{1}{3}r^3\right)\Big|_0^1\cos\theta - \left(\tfrac{1}{2}r^4\right)\Big|_0^1\sin\theta\cos\theta\right]d\theta$$

$$= \int_0^{2\pi}\left[\tfrac{1}{3}\cos\theta - \tfrac{1}{2}\sin\theta\cos\theta\right]d\theta$$

$$= \left(\tfrac{1}{3}\sin\theta - \tfrac{1}{4}\sin^2\theta\right)\Big|_0^{2\pi}$$

$$= \left(\tfrac{1}{3}\cancel{\sin(2\pi)}^{0} - \tfrac{1}{4}\cancel{\sin^2(2\pi)}^{0}\right)$$

$$- \left(\tfrac{1}{3}\cancel{\sin(0)}^{0} - \tfrac{1}{4}\cancel{\sin^2(0)}^{0}\right) = 0 .$$