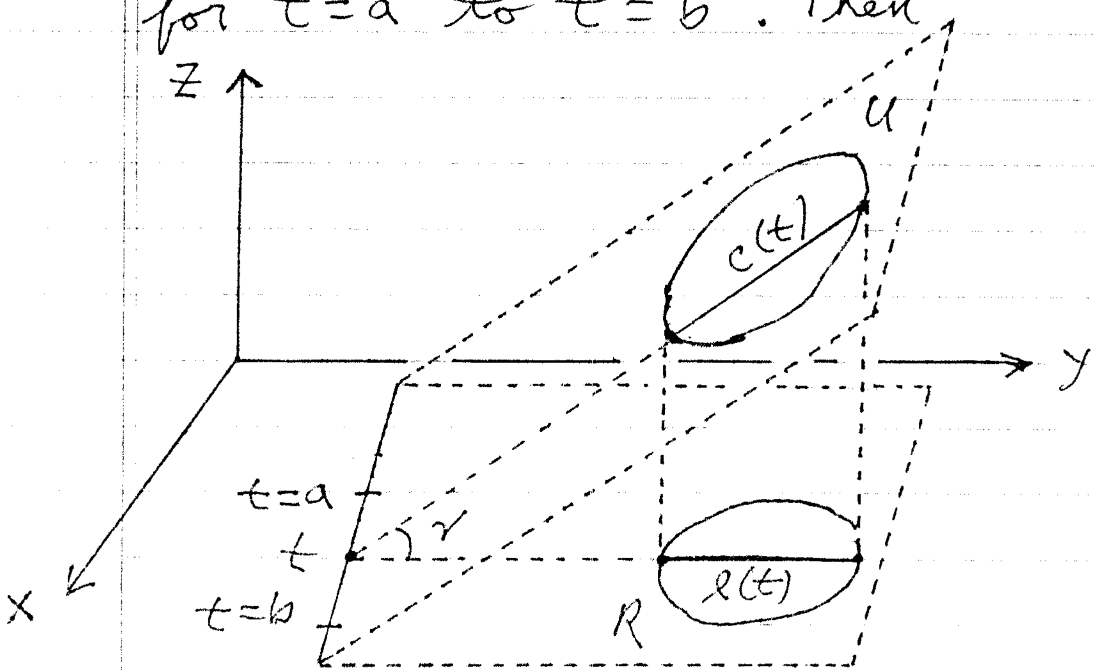
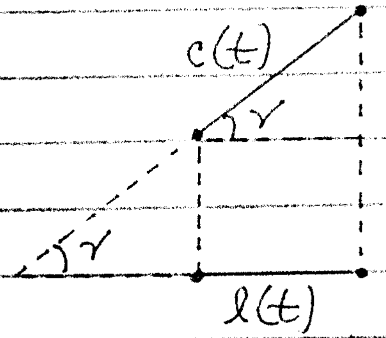
Math 21D
Kouba
Surface Integrals

<u>Theorem A</u>: Let U be a region in a plane that is inclined at an angle $\gamma < \frac{\pi}{2}$ to the xy-plane. Let R be the projection of U onto the xy-plane. Then

$$\text{Area of } U = (\sec \gamma)(\text{Area of } R)$$

<u>proof</u>: Construct a t-axis along the line of intersection of the two planes. Let $c(t)$ be the cross-sectional length of U and let $\ell(t)$ be the cross-sectional length of R for $t=a$ to $t=b$. Then





$$\cos\gamma = \frac{l(t)}{c(t)} \rightarrow$$

$$(\cos\gamma)\, c(t) = l(t),$$

so that

$$\text{Area } R = \int_a^b l(t)\,dt = \int_a^b (\cos\gamma)\, c(t)\,dt$$

$$= (\cos\gamma)\int_a^b c(t)\,dt$$

$$= (\cos\gamma)\cdot(\text{Area } U) \rightarrow$$

$$\text{Area } U = (\sec\gamma)(\text{Area } R) .$$

Theorem: assume that surface S in 3D-space is given by the level surface $f(x,y,z) = c$. Let $\gamma < \pi/2$ be the angle between the tangent plane at point $P = (x,y,z)$ on S and the xy-plane. Then

$$\sec\gamma = \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \vec{k}|} = \frac{\sqrt{(f_x)^2 + (f_y)^2 + (f_z)^2}}{|f_z|} .$$

proof: From Math 21C we know that $\vec{\nabla} f = (f_x)\vec{i} + (f_y)\vec{j} + (f_z)\vec{k}$ is $\perp$ to S at point $P = (x,y,z)$ and $\vec{k}$ is $\perp$ to the xy-plane. The angle between the tangent plane and the XY-plane is the same as the angle between their normal vectors, i.e., for $\gamma < \frac{\pi}{2}$

$$\cos\gamma = \frac{|\vec{\nabla} f \cdot \vec{k}|}{|\vec{\nabla} f|\,\underbrace{|\vec{k}|}_{1}} = \frac{|f_z|}{\sqrt{(f_x)^2+(f_y)^2+(f_z)^2}}$$

so that

$$\sec\gamma = \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \vec{k}|} = \frac{\sqrt{(f_x)^2+(f_y)^2+(f_z)^2}}{|f_z|} .$$

__Def__: Let S be a surface in 3D-space. Partition S into n pieces $S_1, S_2, \ldots, S_n$ each of area $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$. Let $P_i = (x_i, y_i, z_i)$ be a sampling point in S_i for $i = 1, 2, 3, \ldots, n$ and assume function g is defined at all points P in surface S. Then the __surface integral of g over S__ is given by

$$\iint_S g(P)\, dS = \lim_{mesh \to 0} \sum_{i=1}^{n} g(P_i) \cdot \Delta S_i .$$

<u>Method of Computation</u> :

$$\iint_{S} g(P)\,dS = \lim_{mesh \to 0} \sum_{i=1}^{n} g(P_i)\cdot \Delta S_i$$

$$= \lim_{mesh \to 0} \sum_{i=1}^{n} g(P_i)\cdot \sec\gamma \cdot \Delta A_i$$

(Let A_i be the projection of S_i on the XY-plane, where ΔA_i is the area of A_i for $i = 1, 2, 3, \ldots, n$. Apply Theorem A. Assume R is the projection of S onto the XY-plane.)

$$= \iint_{R} g(P)\cdot \sec\gamma \cdot dA \text{ , i.e.,}$$

$$\iint_{S} g(P)\,dS = \iint_{R} g(P)\cdot \sec\gamma \cdot dA \text{ .}$$

Applications of Surface Integrals:

Let $\mathcal{S}$ be a surface and assume the density at point $P = (x, y, z)$ on $\mathcal{S}$ is $\delta(P)$ (mass/area units).

1.) Area $\mathcal{S} = \iint_{\mathcal{S}} 1\, dS$

2.) Mass $\mathcal{S} = \iint_{\mathcal{S}} \delta(P)\, dS$

3.) Moments about planes:

a.) $M_{x=a} = \iint_{\mathcal{S}} (x-a)\,\delta(P)\, dS$

b.) $M_{y=a} = \iint_{\mathcal{S}} (y-a)\,\delta(P)\, dS$

c.) $M_{z=a} = \iint_{\mathcal{S}} (z-a)\,\delta(P)\, dS$

4.) Center of Mass $(\overline{x}, \overline{y}, \overline{z})$:

$$\overline{x} = \frac{\iint_{\mathcal{S}} x\,\delta(P)\, dS}{\iint_{\mathcal{S}} \delta(P)\, dS}, \quad \overline{y} = \frac{\iint_{\mathcal{S}} y\,\delta(P)\, dS}{\iint_{\mathcal{S}} \delta(P)\, dS}, \quad \overline{z} = \frac{\iint_{\mathcal{S}} z\,\delta(P)\, dS}{\iint_{\mathcal{S}} \delta(P)\, dS}$$

5.) Centroid $(\overline{x}, \overline{y}, \overline{z})$:

$$\overline{x} = \frac{\iint_{\mathcal{S}} x\, dS}{\iint_{\mathcal{S}} 1\, dS}, \quad \overline{y} = \frac{\iint_{\mathcal{S}} y\, dS}{\iint_{\mathcal{S}} 1\, dS}, \quad \overline{z} = \frac{\iint_{\mathcal{S}} z\, dS}{\iint_{\mathcal{S}} 1\, dS}$$

6.) Moment of Inertia :

$$M. \text{ of } I. = \iint_{S} (\text{distance})^2 \, \delta(P) \, dS ,$$

where distance refers to the distance from point $P = (x, y, z)$ on surface S to either a point or axis of rotation.