Section 16.5
Thomas Calculus
11th Ed.

Surface Integrals and Flux in 3D-Space

Definition: The Flux of vector field

$\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$

across Surface S in the direction of $\vec{n}$ is

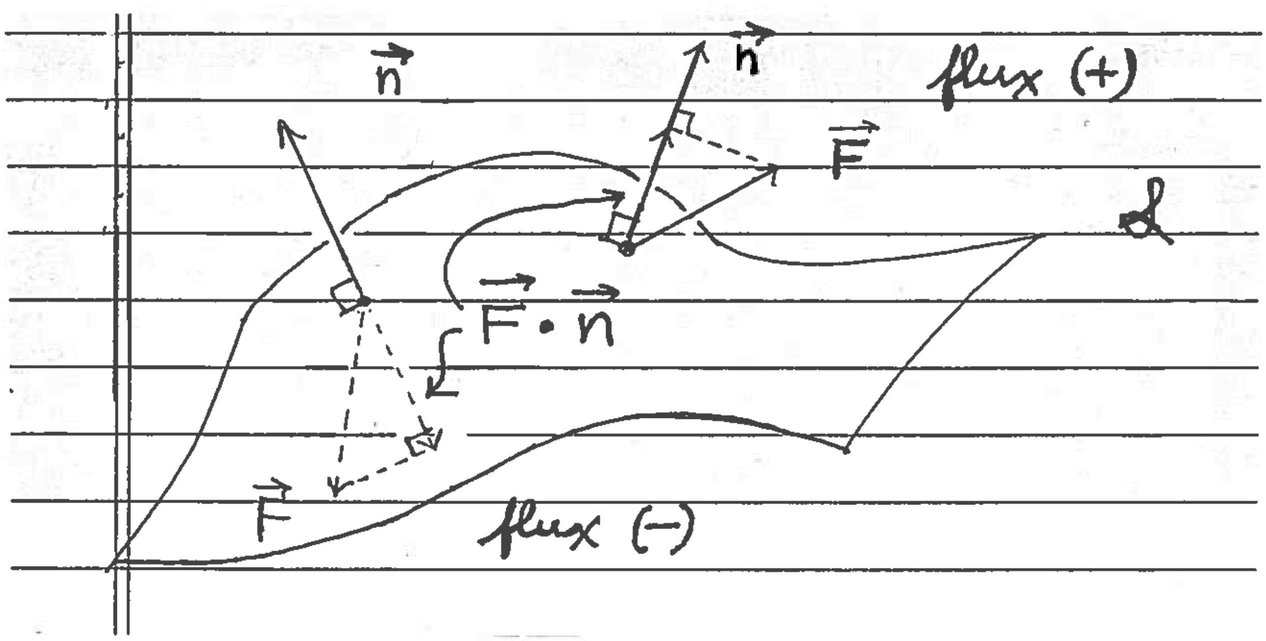
$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n} \; dS \; .$$

Remarks: 1.) Vector $\vec{n}$ is chosen to be normal to ($\perp$) the surface S. If S is given by $f(x,y,z) = c$, then $\vec{n} = \dfrac{\pm \vec{\nabla} f}{|\vec{\nabla} f|}$.

Recall that the component of $\vec{F}$ in the direction of $\vec{n}$ is

$$(\vec{F} \bullet \vec{n})\,\vec{n} \quad ,$$

and $\vec{F} \bullet \vec{n}$ is (+) if the angle θ between $\vec{F}$ and $\vec{n}$ is $0 \le \theta < \frac{\pi}{2}$; $\vec{F} \bullet \vec{n}$ is (−) if the angle θ between $\vec{F}$ and $\vec{n}$ is $\frac{\pi}{2} < \theta \le \pi$.



2.) If $\vec{F} = \delta \cdot v(x,y,z)$

density of fluid: $\frac{\text{mass}}{\text{volume}}$ units

velocity at point (x,y,z): $\frac{\text{length}}{\text{time}}$ units

$$\left(\frac{\text{mass}}{\text{volume}}\right)\left(\frac{\text{length}}{\text{time}}\right) = \frac{\text{mass}}{(\text{area})(\text{time})},$$

then units for

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, dS \text{ are}$$

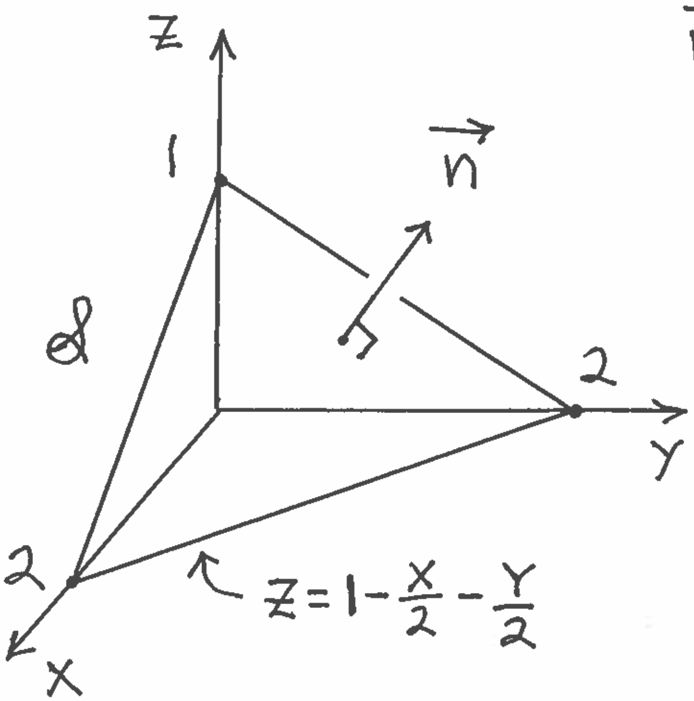
$$\frac{\text{mass}}{(\text{area})(\text{time})} \cdot (\text{area}) = \frac{\text{mass}}{\text{time}}.$$

Example: Assume the velocity vector field is given by

$$\vec{F}(x,y,z) = (y)\vec{i} - (x)\vec{j} + (y+z)\vec{k}$$

(units: $\frac{gm}{(cm.)^2(sec.)}$) and surface S

is the portion of the plane $\frac{x}{2}+\frac{y}{2}+z=1$ in the first octant, with $\vec{n}$ pointing away from the origin.



$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$$

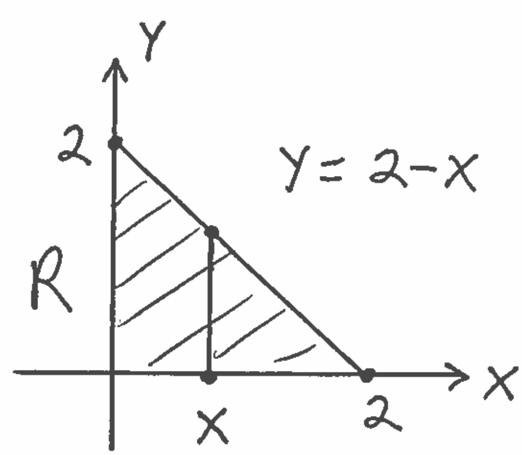
$$= \frac{\left(\frac{1}{2}\right)\vec{i} + \left(\frac{1}{2}\right)\vec{j} + (1)\vec{k}}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (1)^2}}$$

$$= \frac{\left(\frac{1}{2}\right)\vec{i} + \left(\frac{1}{2}\right)\vec{j} + (1)\vec{k}}{\sqrt{\frac{6}{4}}}$$

$$= \frac{2}{\sqrt{6}}\left(\left(\frac{1}{2}\right)\vec{i} + \left(\frac{1}{2}\right)\vec{j} + (1)\vec{k}\right) = \left(\frac{1}{\sqrt{6}}\right)\vec{i} + \left(\frac{1}{\sqrt{6}}\right)\vec{j} + \left(\frac{2}{\sqrt{6}}\right)\vec{k}\ ;$$

then

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, dS$$



$$= \iint_{\mathcal{S}} \left[\frac{1}{\sqrt{6}} y - \frac{1}{\sqrt{6}} x + \frac{2}{\sqrt{6}} (y+z) \right] dS$$

$$= \iint_{\mathcal{S}} \left[\frac{1}{\sqrt{6}} y - \frac{1}{\sqrt{6}} x + \frac{2}{\sqrt{6}} \left(y + 1 - \frac{1}{2} x - \frac{1}{2} y \right) \right] dS$$

$$= \iint_{\mathcal{S}} \left[\cancel{\frac{1}{\sqrt{6}} y} - \frac{1}{\sqrt{6}} x + \frac{2}{\sqrt{6}} y + \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} x - \cancel{\frac{1}{\sqrt{6}} y} \right] dS$$

$$= \iint_{R} \left(\frac{2}{\sqrt{6}} y - \frac{2}{\sqrt{6}} x + \frac{2}{\sqrt{6}} \right) \cdot \frac{|\vec{\nabla} f|}{|f_z|} \, dA$$

$$= \int_0^2 \int_0^{2-x} \left(\frac{2}{\sqrt{6}} y - \frac{2}{\sqrt{6}} x + \frac{2}{\sqrt{6}} \right) \cdot \frac{\sqrt{6}}{2} \, dy \, dx$$

$$= \int_0^2 \int_0^{2-x} (y - x + 1) \, dy \, dx$$

$$= \int_0^2 \left(\frac{1}{2} y^2 - xy + y \right) \Big|_{y=0}^{y=2-x} dx$$

$$= \int_0^2 \left[\frac{1}{2}(2-x)^2 - x(2-x) + (2-x) \right] dx$$

$$= \int_0^2 \left[\frac{1}{2}(4 - 4x + x^2) - 2x + x^2 + 2 - x \right] dx$$

$$= \int_0^2 \left[2 - 2x + \tfrac{1}{2}x^2 - 2x + x^2 + 2 - x\right] dx$$

$$= \int_0^2 \left(\tfrac{3}{2}x^2 - 5x + 4\right) dx$$

$$= \left(\tfrac{1}{2}x^3 - \tfrac{5}{2}x^2 + 4x\right)\Big|_0^2$$

$$= \frac{8}{2} - \frac{20}{2} + \frac{16}{2}$$

$$= \frac{4}{2} = 2 \ \frac{gm.}{sec.}$$