Section 16.5
Thomas Calculus
11th Ed.

Surface Integrals

We have integrated scalar functions over

1.) intervals $[a,b]$: $\int_a^b f(x)\,dx$

2.) regions R in 2D-Space:

$$\iint_R f(x,y)\,dy\,dx$$

3.) regions R in 3D-Space:

$$\iiint_R f(x,y,z)\,dz\,dy\,dx$$

4.) paths C in 2D-Space:

$$\int_C f(x,y)\,ds$$

5.) paths C in 3D-Space:

$$\int_C f(x,y,z)\,ds$$

Now we are going to learn how to integrate a scalar function over a surface $\mathcal{S}$ in 3D-Space, called a Surface Integral, and denoted by

$$\iint_{\mathcal{S}} g(P)\, dS$$

↑ (capital S).

Let's begin with some ground work.

Recall: If $w = f(x, y, z)$ is a scalar function defined on some region in 3D-Space and c is a constant, then we call

$$c = f(x, y, z)$$

a level surface for function f.

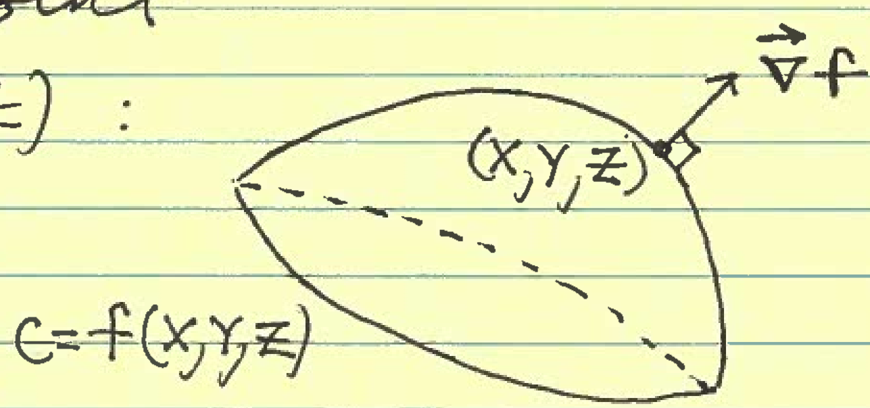
Example: Let $w = x^2 + y^2 + z^2$ and let $w = 1, 4, 9$. Then the level surfaces are all spheres centered at the origin:

$$x^2 + y^2 + z^2 = 1 \quad \text{(radius 1)},$$
$$x^2 + y^2 + z^2 = 4 \quad \text{(radius 2)},$$
$$x^2 + y^2 + z^2 = 9 \quad \text{(radius 3)}$$

Recall: If $w = f(x, y, z)$ and $c = f(x, y, z)$ is a level surface, then the gradient vector

$$\vec{\nabla} f(x, y, z) = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$$

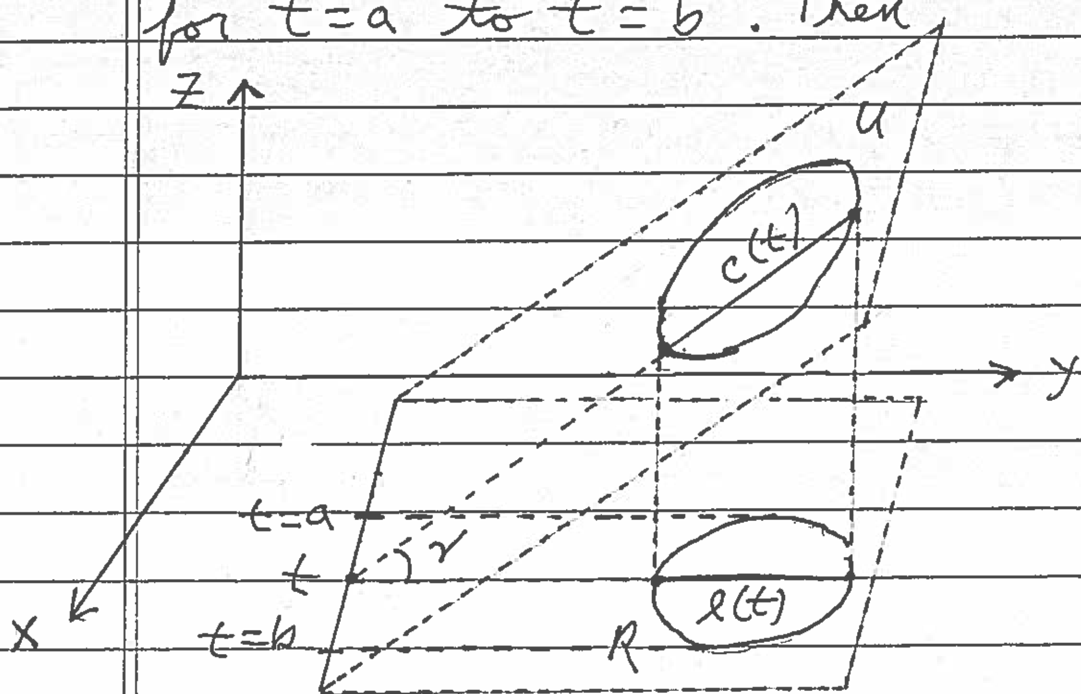
is orthogonal to ($\perp$) the level surface at point (x, y, z):

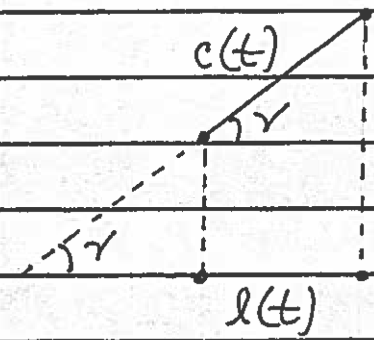


Theorem A: Let U be a region in a plane that is inclined at an angle $\gamma < \frac{\pi}{2}$ to the xy-plane. Let R be the projection of U onto the xy-plane. Then

$$\text{Area of } U = (\sec \gamma)(\text{Area of } R)$$

proof: Construct a t-axis along the line of intersection of the two planes. Let $c(t)$ be the cross-sectional length of U and let $l(t)$ be the cross-sectional length of R for $t = a$ to $t = b$. Then





$$\cos\gamma = \frac{l(t)}{c(t)} \rightarrow$$

$$(\cos\gamma)\, c(t) = l(t),$$

so that

$$\text{Area } R = \int_a^b l(t)\,dt = \int_a^b (\cos\gamma)\, c(t)\,dt$$

$$= (\cos\gamma)\int_a^b c(t)\,dt$$

$$= (\cos\gamma)\cdot(\text{Area } U) \rightarrow$$

$$\text{Area } U = (\sec\gamma)(\text{Area } R).$$

<u>Theorem</u>: Assume that surface S in 3D-space is given by the level surface $f(x,y,z) = c$. Let $\gamma < \pi/2$ be the angle between the tangent plane at point $P = (x,y,z)$ on S and the xy-plane. Then

$$\sec\gamma = \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \vec{k}|} = \frac{\sqrt{(f_x)^2 + (f_y)^2 + (f_z)^2}}{|f_z|}.$$

<u>proof</u>: From Math 21C we know that $\vec{\nabla} f = (f_x)\vec{i} + (f_y)\vec{j} + (f_z)\vec{k}$ is $\perp$ to S at point $P = (x,y,z)$ and $\vec{k}$ is $\perp$ to the xy-plane. The angle

between the tangent plane and the xy-plane is the same as the angle between their normal vectors, i.e., for $\gamma < \frac{\pi}{2}$

$$\cos\gamma = \frac{|\vec{\nabla} f \cdot \vec{k}|}{|\vec{\nabla} f| \underbrace{|\vec{k}|}_{1}} = \frac{|f_z|}{\sqrt{(f_x)^2 + (f_y)^2 + (f_z)^2}}$$

so that

$$\sec\gamma = \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \vec{k}|} = \frac{\sqrt{(f_x)^2 + (f_y)^2 + (f_z)^2}}{|f_z|}.$$

<u>Def</u>: Let $\mathcal{S}$ be a surface in 3D-space. Partition $\mathcal{S}$ into n pieces $S_1, S_2, \ldots, S_n$ each of area $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$. Let $P_i = (x_i, y_i, z_i)$ be a sampling point in S_i for $i = 1, 2, 3, \ldots, n$ and assume function g is defined at all points P in surface $\mathcal{S}$. Then the <u>surface integral of g over $\mathcal{S}$</u> is given by

$$\iint_{\mathcal{S}} g(P)\, dS = \lim_{mesh \to 0} \sum_{i=1}^{n} g(P_i) \cdot \Delta S_i .$$

Method of Computation :

$$\iint_{S} g(P)\, dS = \lim_{\text{mesh}\to 0} \sum_{i=1}^{n} g(P_i)\cdot \Delta S_i$$

$$= \lim_{\text{mesh}\to 0} \sum_{i=1}^{n} g(P_i)\cdot \sec\gamma \cdot \Delta A_i$$

(Let A_i be the projection of S_i on the XY-plane, where ΔA_i is the area of A_i for $i = 1, 2, 3, \ldots, n$. Apply Theorem A. Assume R is the projection of S onto the XY-plane.)

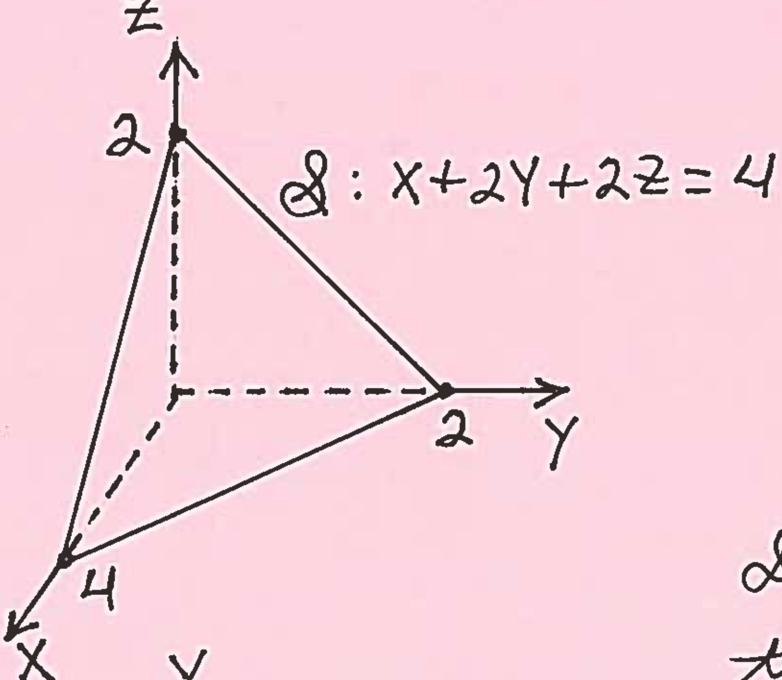
$$= \iint_{R} g(P)\cdot \sec\gamma \cdot dA \text{ , i.e.,}$$

$$\iint_{S} g(P)\, dS = \iint_{R} g(P)\cdot \sec\gamma \cdot dA \; .$$

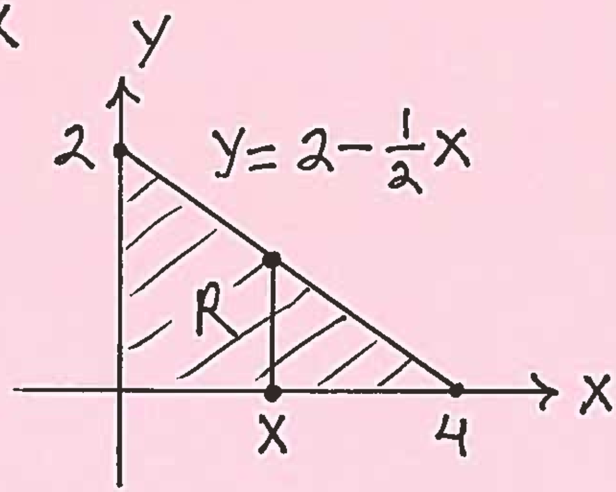
$$= \iint_{R} g(P)\cdot \frac{\sqrt{(f_x)^2 + (f_y)^2 + (f_z)^2}}{|f_z|}\, dA$$

<u>Example</u> : Consider a thin plate S lying on the portion of the plane $x + 2y + 2z = 4$ which is in the first octant. Assume the density at point (x, y, z) is given by

$$\delta(x,y,z) = x + 2y \ \frac{\text{gm.}}{\text{cm.}^2}.$$



Use a Surface Integral to find the Total Mass of the plate.



$$R : \begin{cases} 0 \le x \le 4 \\ 0 \le y \le 2 - \frac{1}{2}x \end{cases}$$

$$\text{Mass} = \iint_S \underset{\uparrow \atop \text{gm.}/\text{cm.}^2}{\delta(x,y,z)} \ \underset{\uparrow \atop \text{cm.}^2}{dS} \quad ;$$

Let's set up our Surface Integral:

$\mathcal{S}: \underbrace{x+2y+2z}_{f(x,y,z)} = 4$, so

$$\text{Mass} = \iint_{\mathcal{S}} \delta(x,y,z)\, dS$$

$$= \iint_{R} \delta(x,y,z) \cdot \frac{\sqrt{(f_x)^2+(f_y)^2+(f_z)^2}}{|f_z|}\, dA$$

$$= \int_0^4 \int_0^{2-\frac{1}{2}x} (x+2y) \frac{\sqrt{(1)^2+(2)^2+(2)^2}}{(2)}\, dy\, dx$$

$$= \frac{3}{2} \int_0^4 (xy+y^2) \Big|_{y=0}^{y=2-\frac{1}{2}x} dx$$

$$= \frac{3}{2} \int_0^4 \left[x\left(2-\tfrac{1}{2}x\right) + \left(2-\tfrac{1}{2}x\right)^2 \right] dx$$

$$= \frac{3}{2} \int_0^4 \left[\cancel{2x} - \tfrac{1}{2}x^2 + 4 - \cancel{2x} + \tfrac{1}{4}x^2 \right] dx$$

$$= \frac{3}{2} \int_0^4 \left(4 - \tfrac{1}{4}x^2\right) dx$$

$$= \frac{3}{2}\left(4x - \frac{1}{12}x^3\right)\Big|_0^4$$

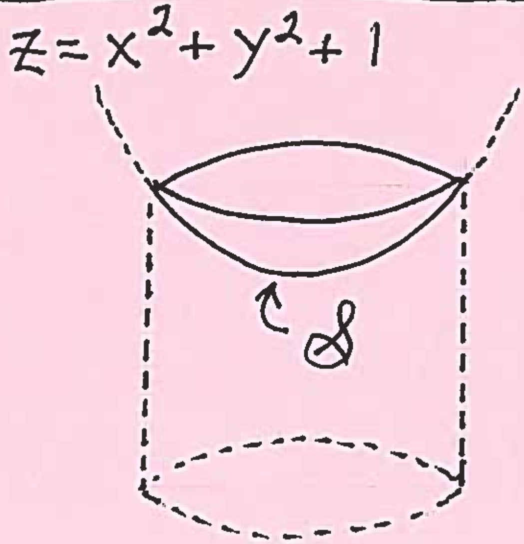
$$= \frac{3}{2}\left(16 - \frac{64}{12}\right) = \frac{3}{4}\left(\frac{48}{3} - \frac{16}{3}\right)$$

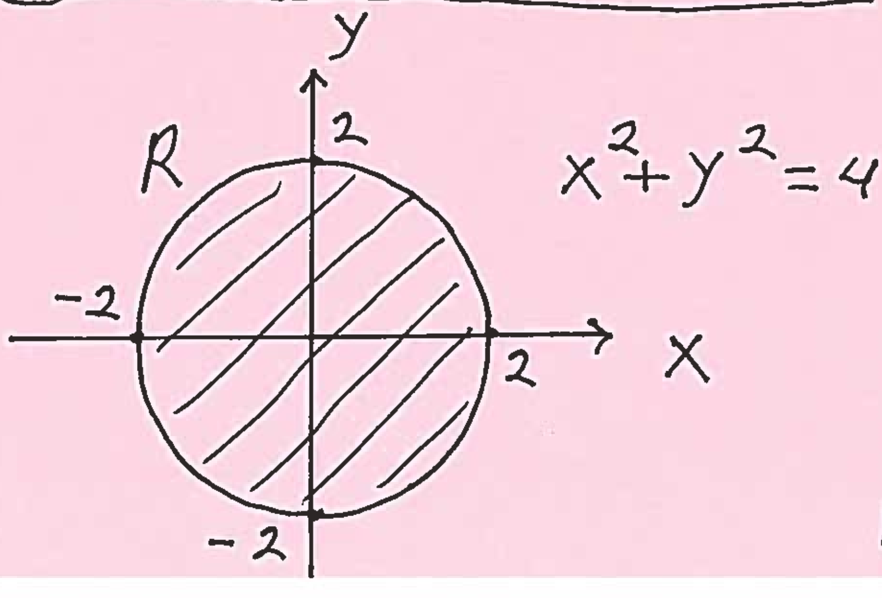
$$= \frac{3}{2} \cdot \frac{32}{3} = 16 \text{ gm.}$$

Example: Evaluate the following Surface Integral

$$\iint_S (xy+1)\, dS,$$

where surface S is that portion of the paraboloid $z = x^2 + y^2 + 1$ inside the cylinder $x^2 + y^2 = 4$.





$$z = x^2 + y^2 + 1 \longrightarrow \underbrace{x^2 + y^2 - z}_{f(x,y,z)} = -1 \, ;$$

then

$$\iint_{S} (xy+1)\, dS = \iint_{R} (xy+1) \frac{\sqrt{(f_x)^2 + (f_y)^2 + (f_z)^2}}{|f_z|}\, dA$$

$$= \iint_{R} (xy+1) \frac{\sqrt{(2x)^2 + (2y)^2 + (-1)^2}}{|-1|}\, dA$$

$$= \iint_{R} (xy+1) \sqrt{4x^2 + 4y^2 + 1}\, dA$$

(Let's try polar coordinates.)

$$R: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \, ; \; x^2 + y^2 = r^2) \end{cases}$$

$$= \iint_{R} (xy+1) \sqrt{4(x^2 + y^2) + 1}\, dA$$

$$= \int_0^{2\pi} \int_0^{2} \big((r\cos\theta)(r\sin\theta) + 1\big) \sqrt{4r^2+1} \cdot r\, dr\, d\theta$$

(This is gonna be a bit challenging.)

$$= \int_0^{2\pi} \int_0^{2} \cos\theta \sin\theta \cdot r^3 \sqrt{4r^2+1} \cdot dr\, d\theta$$

$$+ \int_0^{2\pi} \int_0^{2} r \sqrt{4r^2+1}\, dr\, d\theta$$

$$= A + B \quad ;$$

$$B = \int_0^{2\pi} \int_0^{2} r \left(4r^2+1\right)^{1/2} dr\, d\theta$$

$$= \int_0^{2\pi} \frac{2}{3} \cdot \frac{1}{8} \left(4r^2+1\right)^{3/2} \Big|_{r=0}^{r=2} \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{12} (17)^{3/2} - \frac{1}{12} \right) d\theta$$

$$= \left(\frac{1}{12} (17)^{3/2} - \frac{1}{2} \right) \theta \Big|_0^{2\pi}$$

$$= \left(\frac{1}{12} (17)^{3/2} - \frac{1}{2} \right) (2\pi)$$

$$= \left(\frac{1}{6} (17)^{3/2} - 1 \right) \pi \quad ; \text{ now integrate}$$

$$\int_0^2 r^3 \left(4r^2+1\right)^{1/2} dr = \int_0^2 r \cdot r^2 \left(4r^2+1\right)^{1/2} dr$$

(Let $u = 4r^2+1 \xrightarrow{D} du = 8r\,dr \rightarrow$

$\frac{1}{8} du = r\,dr$, and $r^2 = \frac{1}{4}(u-1)$)

$$= \frac{1}{4}\int_{r=0}^{r=2} (u-1)u^{1/2}\,du = \frac{1}{4}\int_{r=0}^{r=2} (u^{3/2} - u^{1/2})\,du$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_{r=0}^{r=2}$$

$$= \left(\frac{1}{10}(4r^2+1)^{5/2} - \frac{1}{6}(4r^2+1)^{3/2}\right)\Big|_0^2$$

$$= \left(\frac{1}{10}(17)^{5/2} - \frac{1}{6}(17)^{3/2}\right) - \left(\frac{1}{10} - \frac{1}{6}\right)$$

$$= \frac{1}{10}(17)^{5/2} - \frac{1}{6}(17)^{3/2} + \frac{1}{15} = \alpha \;;$$

now

$$A = \int_0^{2\pi}\int_0^2 \cos\theta \sin\theta \cdot r^3\sqrt{4r^2+1}\,dr\,d\theta$$

$$= \int_0^{2\pi} \cos\theta \sin\theta \left(\int_0^2 r^3\sqrt{4r^2+1}\,dr\right) d\theta$$

$$= \int_0^{2\pi} \cos\theta \sin\theta \ (\alpha) \ d\theta$$

$$= \alpha \cdot \frac{1}{2} \sin^2\theta \Big|_0^{2\pi}$$

$$= \alpha \cdot \frac{1}{2}\left(\cancelto{0}{\sin^2 2\pi} - \cancelto{0}{\sin^2 0}\right)$$

$= 0$; so finally

$$\iint_S (xy+1)\, dS = A + B$$

$$= 0 + \frac{2}{3}(5)^{3/2} + \frac{2}{15}$$

$$= \frac{2}{3}(5)^{3/2} + \frac{2}{15}$$