Unit Tangent Vector, Unit Normal Vector, Arc Length, Curvature

position vector: \( \vec{r}(t) = f(t) \hat{i} + g(t) \hat{j} + h(t) \hat{k} \)

velocity vector: \( \vec{v}(t) = f'(t) \hat{i} + g'(t) \hat{j} + h'(t) \hat{k} \)

acceleration vector: \( \vec{a}(t) = f''(t) \hat{i} + g''(t) \hat{j} + h''(t) \hat{k} \)

\[ |\vec{v}(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \]

Recall: (Arc Length) Let curve \( C \) be determined by vector function \( \vec{r}(t) \). The arc length \( S \) for \( t = a \) to \( t = b \) is

\[ S = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \, dt \]

\[ = \int_a^b |\vec{v}(t)| \, dt \]
Def: Let $\vec{r}(t)$ be a vector function which plots a curve $C$ in space. The unit tangent vector for $\vec{r}(t)$ is

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

Note: 1.) $\vec{T}(t)$ points in the direction of motion along $C$. 2.) $\vec{T}(t)$ is a unit vector.

Notation:

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{d}{dt} \frac{\vec{r}(t)}{ds}$$

(speed $= \frac{ds}{dt} = |\vec{v}(t)|$)

$$= \frac{d\vec{r}(t)}{dt} \cdot \frac{dt}{ds}$$

(assume time $t = t(s)$ is a function of arc length $s$.)

$$= \frac{d}{ds} \vec{r}(t(s))$$

There are situations where we may want to discuss vector...
function \( \vec{r}(t) = \vec{r}(t(s)) \) as a function of its arc length \( s \).

**Def:** Let \( \vec{r}(t) \) be a vector function and \( \vec{T}(t) \) its unit tangent vector. The principal unit normal vector for \( \vec{r}(t) \) is

\[
\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}.
\]

**Theorem:**

a.) \( \vec{N}(t) \) is a unit vector.

b.) \( \vec{N}(t) \) is normal to the path \( C \) determined by \( \vec{r}(t) \), i.e., \( \vec{N}(t) \) is orthogonal to \( \vec{T}(t) \).

c.) \( \vec{N}(t) \) points in the direction that curve \( C \) is turning.

**Proof:**
a.) Obvious.
b.) \( \vec{T}(t) \) is a unit vector \( \implies |\vec{T}(t)| = 1 \); also \( |\vec{T}(t)|^2 = \vec{T}(t) \cdot \vec{T}(t) = 1^2 = 1 \implies \vec{T}(t) \cdot \vec{T}(t) = 1 \implies \vec{T}'(t) \cdot \vec{T}(t) + \vec{T}(t) \cdot \vec{T}'(t) = 0 \implies \vec{T}'(t) \cdot \vec{T}(t) = 0 \implies \vec{T}'(t) \perp \vec{T}(t) \implies \vec{T}'(t) \perp \vec{T}(t) \implies \vec{N}(t) \perp \vec{T}(t) \).
c. \[ T'(t) = \lim_{h \to 0} \frac{T(t+h) - T(t)}{h} \]
(turning right)

\[
\begin{align*}
\overrightarrow{T}(t) & \quad \overrightarrow{T}(t+h) \\
\overrightarrow{n}(t) & \quad \overrightarrow{n}(t+h) \\
(0,0,0) & \\
\end{align*}
\]

\[
\begin{align*}
\overrightarrow{T}(t) - \overrightarrow{T}(t+h) & \quad \text{points right} \\
\overrightarrow{T}(t+h) - \overrightarrow{T}(t) & \quad \text{points left} \\
\end{align*}
\]

**Curvature**

**Def:** Let \( \overrightarrow{n}(t) \) be a vector function and \( \overrightarrow{T}(t) \) its unit tangent vector. The curvature of the path \( C \) is

\[ \kappa = \left| \frac{d \overrightarrow{T}}{ds} \right| \]
Note that
\[ K = \left| \frac{d\vec{T}}{ds} \right| = \left| \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds} \right| = \left| \vec{T}'(t) \cdot \frac{1}{ds/dt} \right| = \frac{1}{|\vec{V}(t)|} \cdot |\vec{T}'(t)| . \]

**Formula for Computing Curvature:**
\[ K = \frac{1}{|\vec{V}(t)|} \cdot |\vec{T}'(t)| \]

**Fact:** The curvature of a circle of radius \( a \) is \( K = \frac{1}{a} \).

**Def:** The circle of curvature at a point \( P \) on path \( C \) (in 2D-space) is the circle in the plane that

1.) is tangent to the curve at \( P \) (has same tangent line)
2.) has the same curvature that path \( C \) has at \( P \)
3.) lies toward the concave (inner) side of path \( C \)