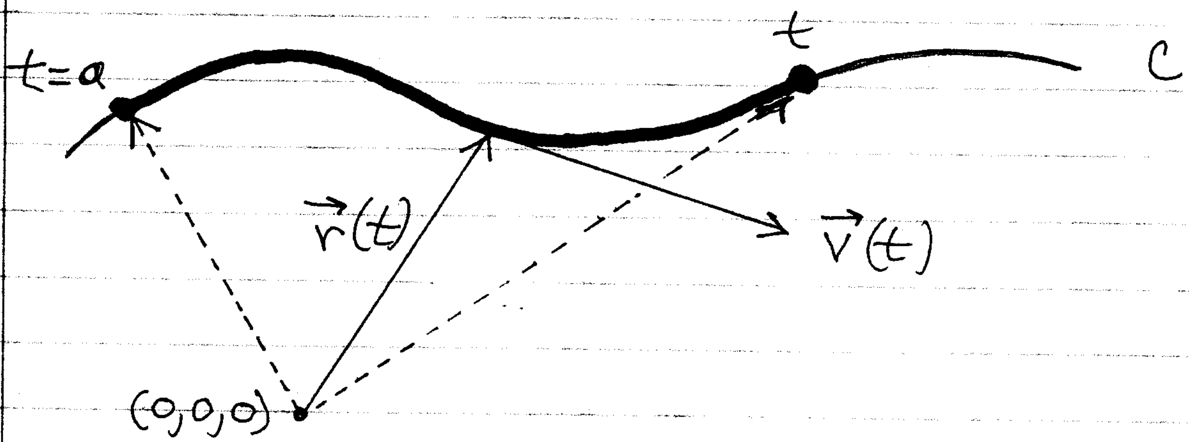
Math 21D
Kouba
Unit Tangent Vector, Unit Normal Vector, Arc Length, Curvature

position vector : $\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$,
velocity vector : $\vec{v}(t) = f'(t)\,\vec{i} + g'(t)\,\vec{j} + h'(t)\,\vec{k}$,
acceleration vector : $\vec{a}(t) = f''(t)\,\vec{i} + g''(t)\,\vec{j} + h''(t)\,\vec{k}$,

$$|\vec{v}(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \; ;$$

<u>Recall</u>: (Arc Length) Let curve C be determined by vector function $\vec{r}(t)$. The arc length s for $t=a$ to t is



$$s = \int_a^t \sqrt{(f'(\tau))^2 + (g'(\tau))^2 + (h'(\tau))^2}\, d\tau$$

$$= \int_a^t |\vec{v}(\tau)|\, d\tau \; .$$

Def: Let $\vec{r}(t)$ be a vector function which plots a curve C in space. The unit tangent vector for $\vec{r}(t)$ is

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

Note: 1.) $\vec{T}(t)$ points in the direction of motion along C.
2.) $\vec{T}(t)$ is a unit vector.

Notation:

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

$$= \frac{\frac{d}{dt}\vec{r}(t)}{\frac{ds}{dt}} \qquad \left(\text{speed} = \frac{ds}{dt} = |\vec{v}(t)|\right)$$

$$= \frac{d\vec{r}(t)}{dt} \cdot \frac{dt}{ds} \qquad \text{(assume time } t = t(s) \text{ is a function of arc length } s.)$$

$$= \frac{d\,\vec{r}(t(s))}{ds}.$$

There are situations where we may want to discuss vector function $\vec{r}(t) = \vec{r}(t(s))$ as a function of its arc length s.

Def: Let $\vec{r}(t)$ be a vector function and $\vec{T}(t)$ its unit tangent vector. The principal unit normal vector for $\vec{r}(t)$ is

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}.$$

Theorem: a.) $\vec{N}(t)$ is a unit vector.

b.) $\vec{N}(t)$ is normal to the path C determined by $\vec{r}(t)$, i.e., $\vec{N}(t)$ is orthogonal to $\vec{T}(t)$.

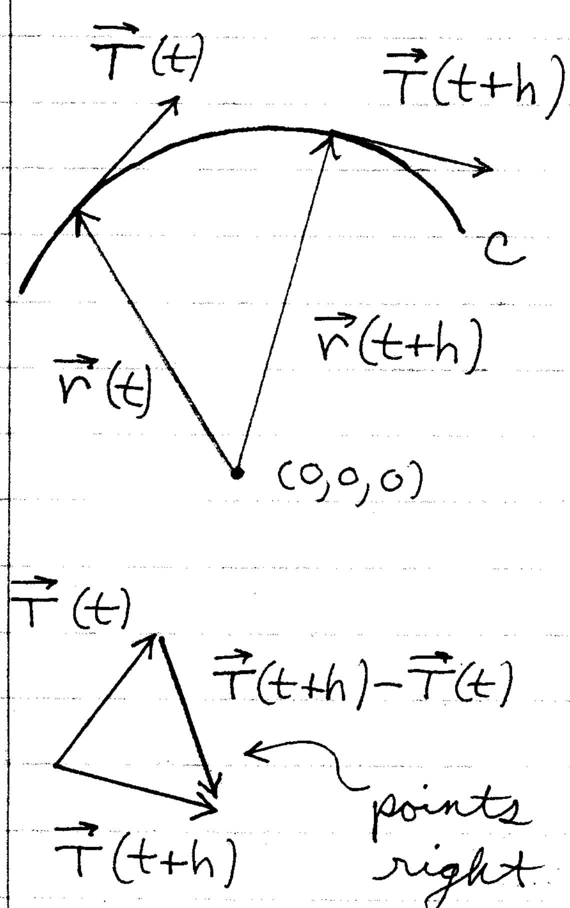
c.) $\vec{N}(t)$ points in the direction that curve C is turning.

Proof: a.) Obvious.

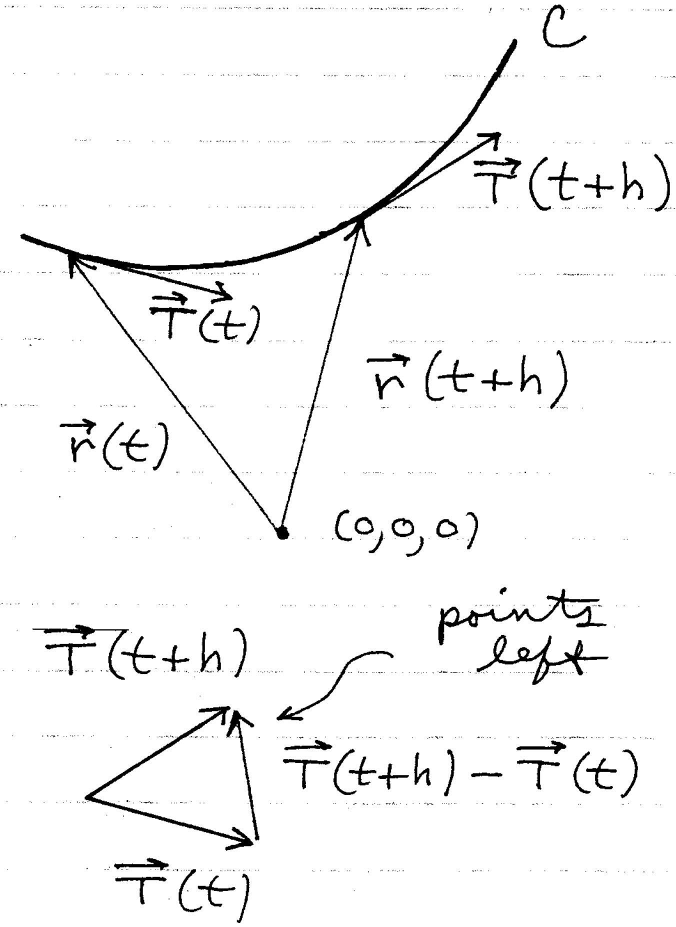
b.) $\vec{T}(t)$ is a unit vector $\rightarrow |\vec{T}(t)| = 1$; also $|\vec{T}(t)|^2 = \vec{T}(t) \cdot \vec{T}(t) = 1^2 = 1 \rightarrow$
$\vec{T}(t) \cdot \vec{T}(t) = 1 \xrightarrow{D} \vec{T}'(t) \cdot \vec{T}(t) + \vec{T}(t) \cdot \vec{T}'(t) = 0$
$\rightarrow 2\vec{T}'(t) \cdot \vec{T}(t) = 0 \rightarrow \vec{T}'(t) \cdot \vec{T}(t) = 0 \rightarrow$
$\vec{T}'(t) \perp \vec{T}(t) \rightarrow \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|} \perp \vec{T}(t) \rightarrow$
$\vec{N}(t) \perp \vec{T}(t)$.

c.) $\vec{T}'(t) = \lim\limits_{h \to 0} \dfrac{\vec{T}(t+h) - \vec{T}(t)}{h}$:

(turning right)



(turning left)



Curvature

<u>Def</u>: Let $\vec{r}(t)$ be a vector function and $\vec{T}(t)$ its unit tangent vector. The <u>curvature</u> of the path C is

$$K = \left| \frac{d\vec{T}}{ds} \right| .$$

Note that

$$K = \left| \frac{d\vec{T}}{dS} \right| = \left| \frac{d\vec{T}}{dt} \cdot \frac{dt}{dS} \right|$$

$$= \left| \vec{T}'(t) \cdot \frac{1}{\frac{dS}{dt}} \right| = \frac{1}{|\vec{V}(t)|} \cdot |\vec{T}'(t)| .$$

Formula for Computing Curvature :

$$K = \frac{1}{|\vec{V}(t)|} \cdot |\vec{T}'(t)|$$

Fact : The curvature of a circle of radius a is $K = 1/a$.

Def : The circle of curvature at a point P on path C (in 2D-space) is the circle in the plane that

1.) is tangent to the curve at P (has same tangent line)
2.) has the same curvature that path C has at P
3.) lies toward the concave (inner) side of path C