Section 15.4
Thomas Calculus
11th Ed.

Triple Integrals Using Rectangular Coordinates

We will use Triple Integrals to compute Volume, Mass, Moments, Moments of Inertia, Centroids, Centers of Mass, etc., of three-dimensional objects.

SEE the following handout for the Definition of a Triple Integral.

Math 21D
Kouba
Triple Integrals Over Solid Regions R Using Rectangular Coordinates

Assume that function $w = f(P)$ is defined on a solid region R in 3D-space, i.e., $w = f(x, y, z)$. Partition R into n parts $R_1, R_2, \ldots, R_n$ of volumes $\Delta V_1, \Delta V_2$, and ΔV_n, resp. Pick sampling points $P_i = (x_i, y_i, z_i)$ in R_i for $i = 1, 2, \ldots, n$. Define the diameter of R_i, $\operatorname{diam}(R_i)$, to be the maximum distance between points in R_i and define the mesh of the partition to be

$$\text{mesh} = \max_{1 \le i \le n} (\operatorname{diam}(R_i)) .$$

Then

$$\iiint_R f(P)\, dV = \lim_{\text{mesh} \to 0} \sum_{i=1}^{n} f(P_i) \cdot \Delta V_i .$$

If $\Delta V_i = (\Delta z_i)(\Delta y_i)(\Delta x_i)$ for $i = 1, 2, \ldots, n$, then

$$\iiint_R f(P)\, dV = \iiint_R f(P)\, dz\, dy\, dx .$$

<u>Facts</u>: 1.) $\iiint_R 1\, dV = \boxed{\text{Volume}}$ of R

2.) $\iiint_R \delta(P)\, dV = \boxed{\text{Mass}}$ of R, where

$\delta(P)$ is density $\left(\frac{\text{mass}}{\text{volume}} \text{ units}\right)$

3.) The $\boxed{\underline{\text{Average}}\ \underline{\text{Value}}}$ of $z = f(P)$ on solid region R is

$$AVE = \frac{1}{\text{volume of } R} \cdot \iiint_R f(P)\, dV$$

Example: Evaluate the following Triple Integrals.

1.) $\int_{-1}^{0}\int_{0}^{2}\int_{1}^{3} 3xe^{y}z^{2}\,dz\,dy\,dx$

$= \int_{-1}^{0}\int_{0}^{2} \left(\cancel{3}xe^{y}\cdot\frac{1}{\cancel{3}}z^{3}\right)\Big|_{z=1}^{z=3} dy\,dx$

$= \int_{-1}^{0}\int_{0}^{2} xe^{y}(27-1)\,dy\,dx$

$= 26\int_{-1}^{0} (xe^{y})\Big|_{y=0}^{y=2} dx$

$= 26\int_{-1}^{0} (xe^{2}-x)\,dx$

$= 26\cdot\left(\frac{1}{2}x^{2}e^{2}-\frac{1}{2}x^{2}\right)\Big|_{x=-1}^{x=0}$

$= 26\cdot\frac{1}{2}\left[(0-0)-(e^{2}-1)\right]$

$= -13(e^{2}-1)$

2.) $$\int_0^{\frac{\pi}{6}} \int_0^{Y} \int_0^{X-Y} 180 \cdot \cos(3X+2Y-Z)\, dZ\, dX\, dY$$

$$= \int_0^{\frac{\pi}{6}} \int_0^{Y} \left(-180 \sin(3X+2Y-Z) \Big|_{Z=0}^{Z=X-Y} \right) dX\, dY$$

$$= \int_0^{\frac{\pi}{6}} \int_0^{Y} \left[-180 \sin(3X+2Y-(X-Y)) - {}^{-}180 \sin(3X+2Y) \right] dX\, dY$$

$$= \int_0^{\frac{\pi}{6}} \int_0^{Y} \left[180 \sin(3X+2Y) - 180 \sin(2X+3Y) \right] dx\, dy$$

$$= \int_0^{\frac{\pi}{6}} \left(-60 \cos(3X+2Y) + 90 \cos(2X+3Y) \right) \Big|_{X=0}^{X=Y} dy$$

$$= \int_0^{\frac{\pi}{6}} \left[\left(-60 \cos(5Y) + 90 \cos(5Y) \right) - \left(-60 \cos(2Y) + 90 \cos(3Y) \right) \right] dy$$

$$= \int_0^{\frac{\pi}{6}} \left(30 \cos(5Y) + 60 \cos(2Y) - 90 \cos(3Y) \right) dy$$

$$= \left[6 \sin(5Y) + 30 \sin(2Y) - 30 \sin(3Y) \right] \Big|_0^{\frac{\pi}{6}}$$

$$= \left(6 \sin\left(\frac{5\pi}{6}\right) + 30 \sin\left(\frac{\pi}{3}\right) - 30 \sin\left(\frac{\pi}{2}\right) \right)$$

$$- \left(6 \underbrace{\sin(0)}_{0} + 30 \underbrace{\sin(0)}_{0} - 30 \underbrace{\sin(0)}_{0} \right)$$

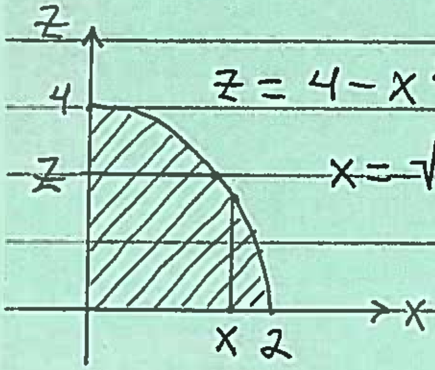
$$= 6 \cdot \left(\frac{1}{2}\right) + 30 \left(\frac{\sqrt{3}}{2}\right) - 30(1)$$

$$= 3 + 15\sqrt{3} - 30 \quad = 15\sqrt{3} - 27$$

3.) $\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z}\, dy\, dz\, dx$

$= \int_0^2 \int_0^{4-x^2} \left(\frac{\sin 2z}{4-z} \cdot y \Big|_{y=0}^{y=x} \right) dz\, dx$

$= \int_0^2 \int_0^{4-x^2} \frac{\sin 2z}{4-z} \cdot x \, dz\, dx \longleftarrow$ can't integrate, so switch order



$z = 4 - x^2$ or $x = \sqrt{4-z}$

$= \int_0^4 \int_0^{\sqrt{4-z}} \frac{\sin 2z}{4-z} \cdot x \, dx\, dz$

$= \int_0^4 \left(\frac{\sin 2z}{4-z} \cdot \frac{x^2}{2} \Big|_{x=0}^{x=\sqrt{4-z}} \right) dz$

$= \int_0^4 \frac{\sin 2z}{\cancel{4-z}} \cdot \frac{\cancel{4-z}}{2}\, dz$

$= \frac{1}{2} \cdot \frac{-1}{2} \cos 2z \Big|_0^4$

$= \frac{-1}{4} \left(\cos 8 - \overset{1}{\cancel{\cos 0}} \right) = \frac{1}{4} (1 - \cos 8)$