Section 15.6
Thomas Calculus
11th Ed.

Triple Integrals Over Solid Regions R in 3D-Space Using Spherical Coordinates

We already know how to plot points in 3D-Space using

Rectangular Coordinates:

(x, y, z)

and

Cylindrical Coordinates

(r, θ, z)

Let's introduce a third coordinate system called

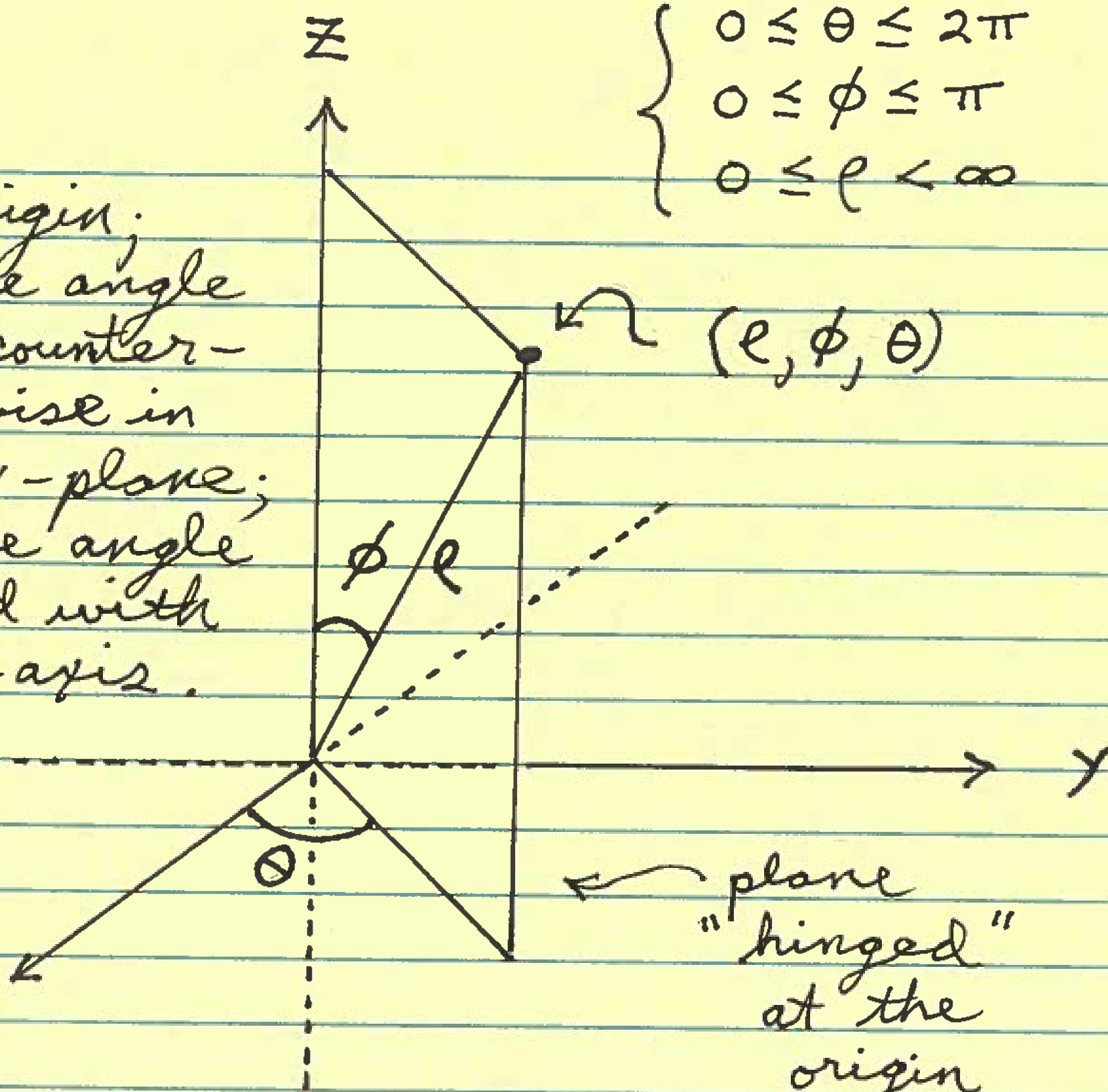
Spherical Coordinates:

It uses TWO angles, θ and ϕ, and ONE distance, ρ;

ρ is the length of a segment starting

$$\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \\ 0 \le \rho < \infty \end{cases}$$

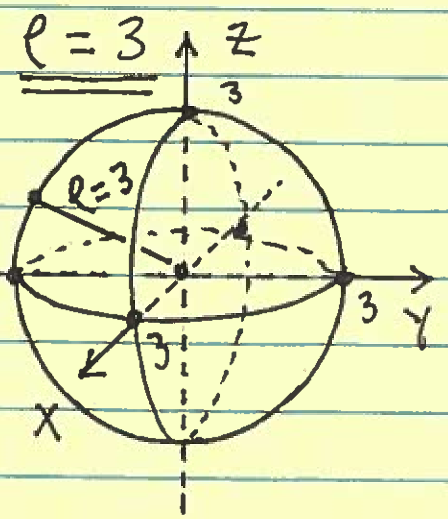
at the origin; θ is the angle swept counter-clockwise in the XY-plane; ϕ is the angle formed with the z-axis.



<u>Example</u>: Sketch the graphs of the following spherical equations in 3D-Space.

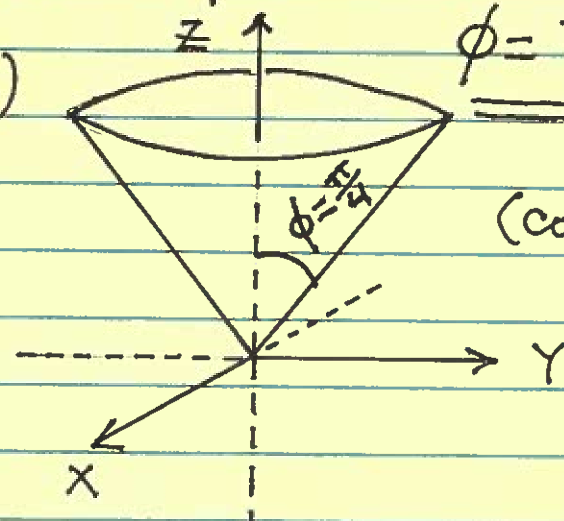
1.) $\underline{\underline{\rho = 3}}$

(sphere of radius 3)

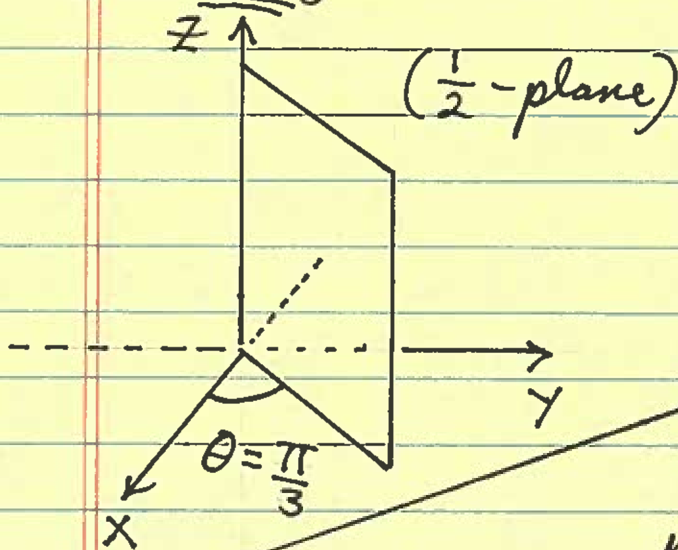


2.) $\underline{\underline{\phi = \frac{\pi}{4}}}$

(cone)

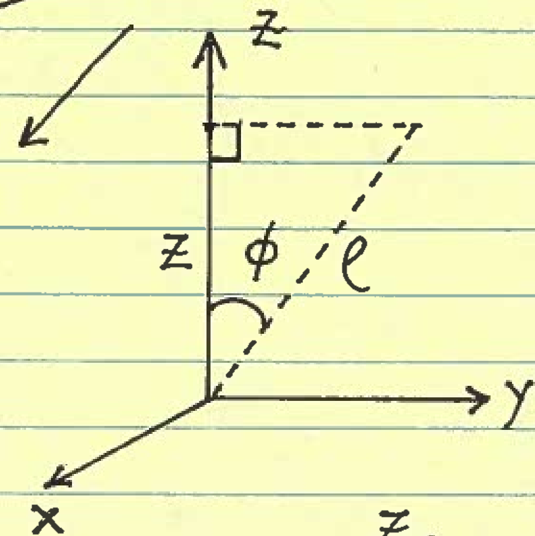


3.) $\underline{\underline{\theta = \frac{\pi}{3}}}$



4.) $\underline{\underline{\rho = 5 \sec \phi}}$

(Hmmm... Let's convert this equation to Rectangular Coordinates:

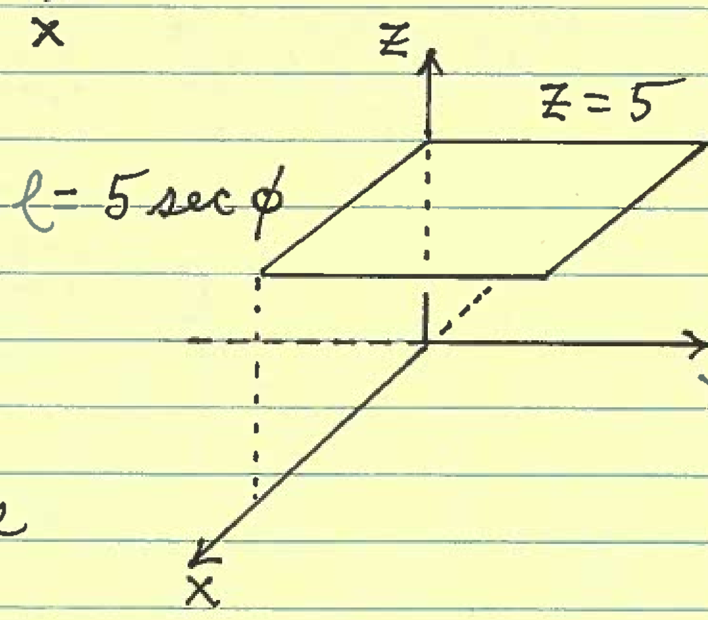


$$\cos \phi = \frac{z}{\rho}$$

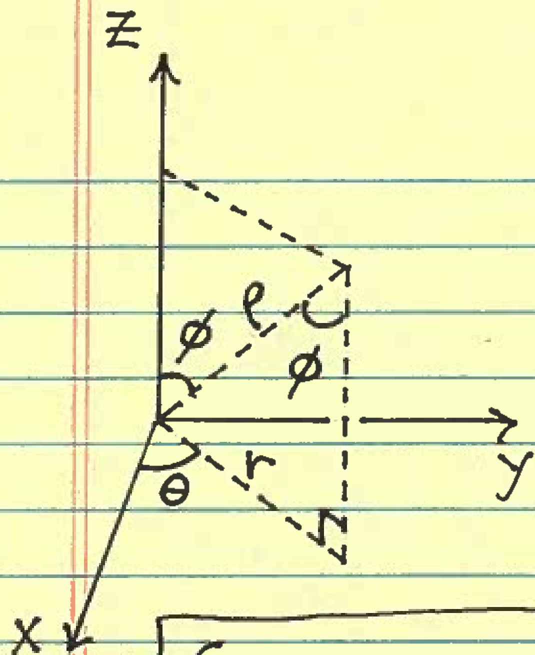
→ $\boxed{z = \rho \cos \phi} = 5$

→ $\underline{\underline{z = 5}}$

a horizontal plane



Let's consider other conversions between coordinate systems:



$$\sin\phi = \frac{r}{\rho} \rightarrow \boxed{r = \rho\sin\phi},$$

then

$$\boxed{\begin{cases} \underline{x = r\cos\theta = \rho\sin\phi\cos\theta} \\ \underline{y = r\sin\theta = \rho\sin\phi\sin\theta} \end{cases}}, \text{ and}$$

$$x^2 + y^2 + z^2 = (\rho\sin\phi\cos\theta)^2$$

$$+ (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2$$

$$= \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta$$

$$+ \rho^2\cos^2\phi$$

$$= \rho^2\sin^2\phi\,(\underbrace{\cos^2\theta + \sin^2\theta}_{1})$$

$$+ \rho^2\cos^2\phi$$

$$= \rho^2(\sin^2\phi + \cos^2\phi)$$

$$= \rho^2(1) = \rho^2, \text{ i.e,}$$

$$\boxed{x^2 + y^2 + z^2 = \rho^2}.$$

Recall that the Differentials of Volume are:

$$\begin{cases} \text{Rect. Coord.}: & dV = dz\,dy\,dx \\ \text{Cyl. Coord.}: & dV = r\,dz\,dr\,d\theta \end{cases}$$

What is the Differential of Volume for Spherical Coordinates?

SEE HANDOUT on NEXT PAGE

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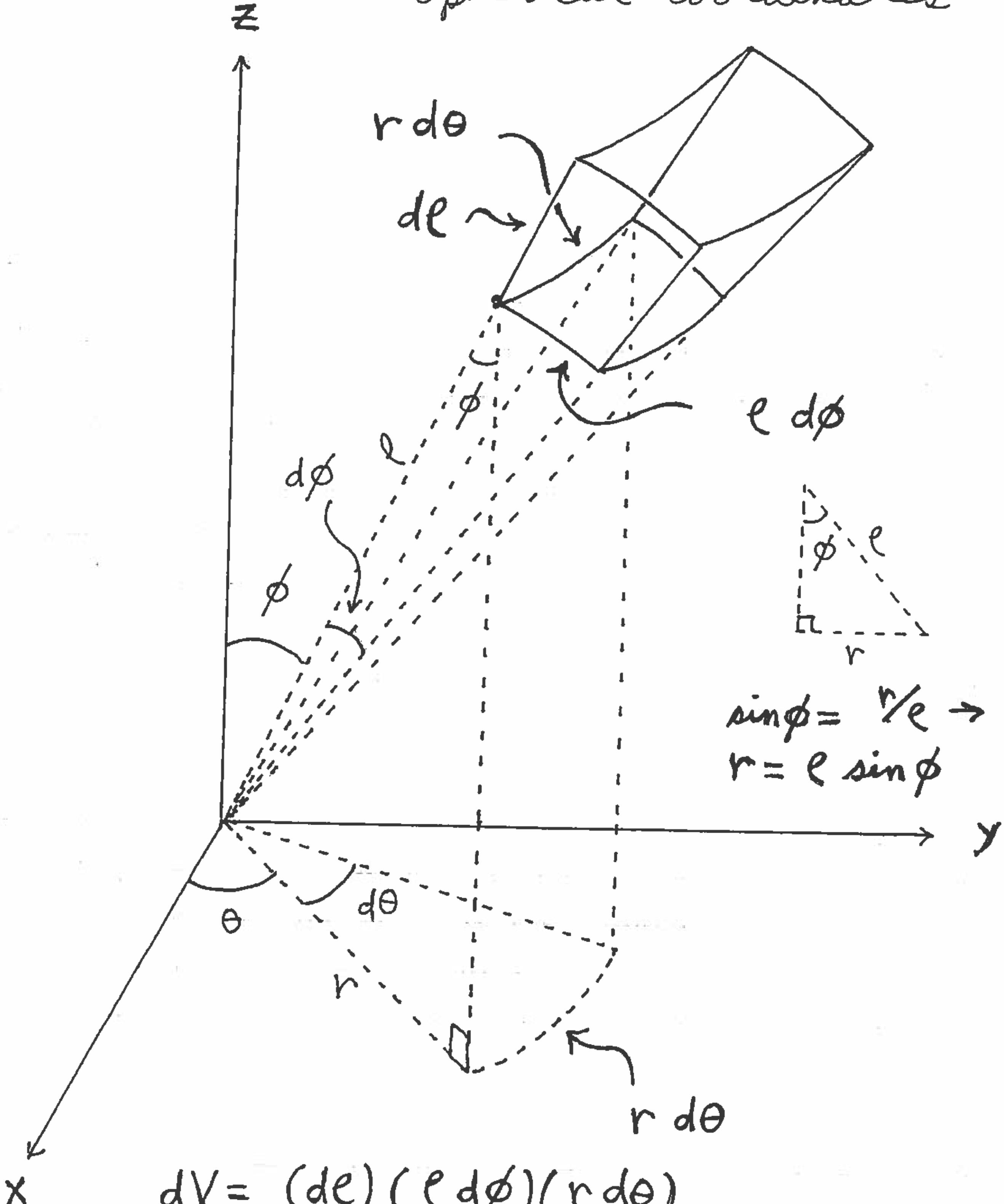
The Differential of Volume for
Spherical Coordinates

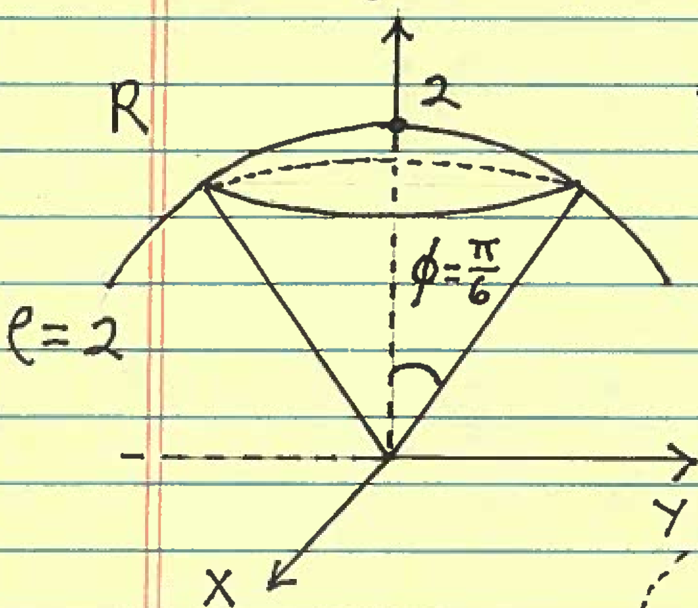


$$dV = (d\rho)(\rho\, d\phi)(r\, d\theta)$$
$$= (d\rho)(\rho\, d\phi)(\rho \sin\phi\, d\theta)$$
$$= \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Example: Consider the solid R inside the cone $\phi = \frac{\pi}{6}$ and below the top half of the sphere $\rho = 2$.

1.) Sketch the surfaces and the solid R.

2.) Find the Volume of R by using Spherical Coordinates.



$$\text{Vol.}\,R = \iiint_R 1\, dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}} \sin\phi \cdot \frac{1}{3}\rho^3 \Big|_{\rho=0}^{\rho=2} d\phi\, d\theta$$

$$R: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{6} \\ 0 \le \rho \le 2 \end{cases}$$

$$= \frac{8}{3}\int_0^{2\pi} -\cos\phi \Big|_{\phi=0}^{\phi=\frac{\pi}{6}} d\theta$$

$$= \frac{8}{3}\int_0^{2\pi}\left(-\overset{-\sqrt{3}/2}{\cancel{\cos}\frac{\pi}{6}} - -\overset{1}{\cancel{\cos}\,0}\right) d\theta$$

$$= \frac{8}{3}\left(1 - \frac{\sqrt{3}}{2}\right)\cdot \theta \Big|_0^{2\pi} = \frac{16}{3}\pi \cdot \left(1 - \frac{\sqrt{3}}{2}\right)$$

Example: Sketch the graph of $\rho = 2\cos\phi$ by first converting to Rectangular Coordinates.

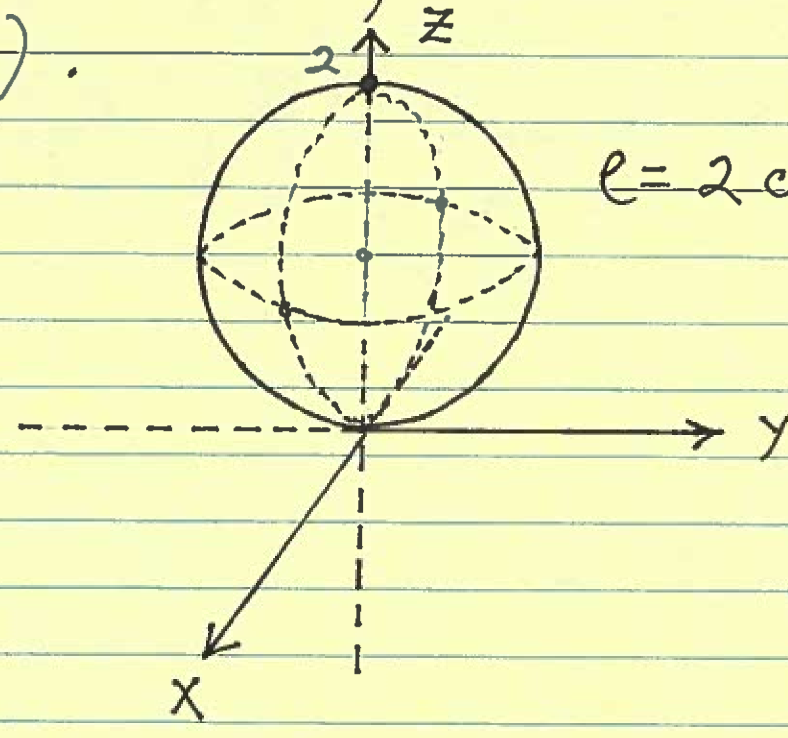
$\rho = 2\cos\phi \rightarrow \rho^2 = 2\rho\cos\phi \rightarrow$

$x^2 + y^2 + z^2 = 2z \rightarrow$

$x^2 + y^2 + z^2 - 2z = 0 \rightarrow$

$x^2 + y^2 + (z^2 - 2z + 1) = 1 \rightarrow$

$\boxed{x^2 + y^2 + (z-1)^2 = 1}$, a sphere of radius $r = 1$, centered at $(0, 0, 1)$.



$\rho = 2\cos\phi$

Example: Sketch solid R in 3D-Space. Its description is given by

$$R: \begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ \frac{\pi}{6} \le \phi \le \frac{\pi}{2} \\ 0 \le \rho \le 3\csc\phi \end{cases}$$

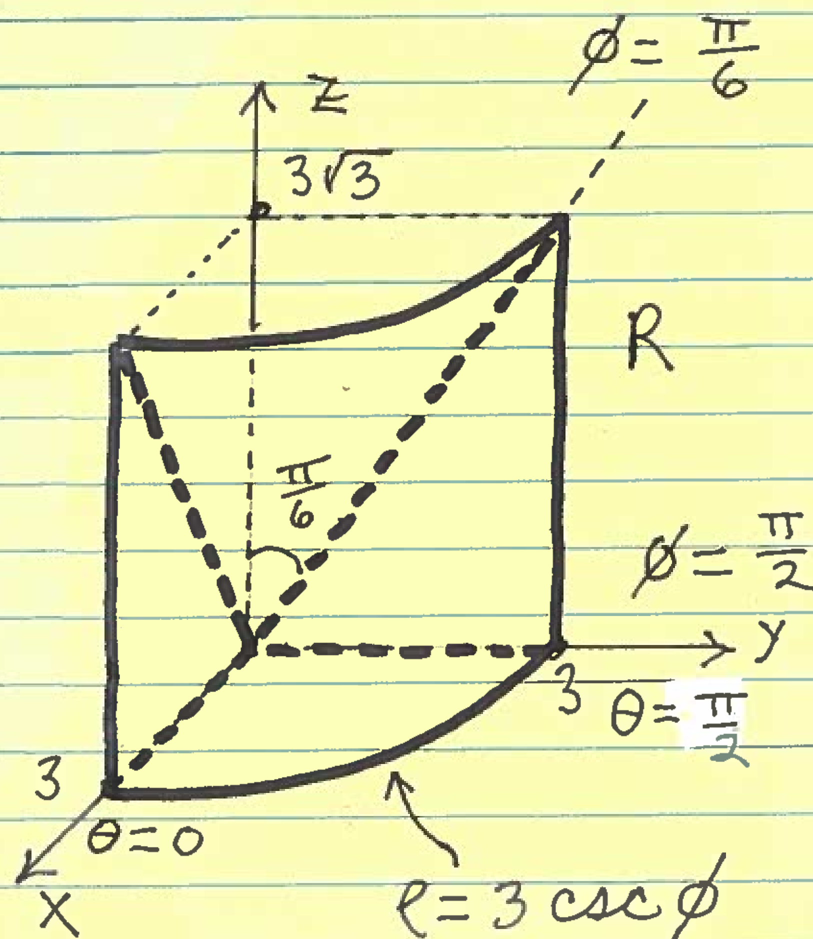
If $\rho = 3\csc\phi$

$\rightarrow \rho = 3 \cdot \frac{1}{\sin\phi}$

$\rightarrow \rho \sin\phi = 3$

$\rightarrow r = 3$ (CYL. COORD.)

a cylinder of radius 3



Example: Consider solid region R inside the sphere given by $\rho = 2\cos\phi$ in the previous example. If density at any point $P = (x, y, z)$ is given by

$$\delta(P) = \delta(x,y,z) = z \ \ \text{gm.}/\text{cm.}^3 ,$$

find the total Mass of solid R.

$$R: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{2} \\ 0 \le \rho \le 2\cos\phi \end{cases} , \text{ then}$$

$$\text{Mass} = \iiint_R \delta(P)\, dV = \iiint_R z\, dV$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\cos\phi} (\rho\cos\phi)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\cos\phi} \rho^3 \cos\phi \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos\phi \sin\phi \cdot \frac{1}{4} \rho^4 \Big|_{\rho=0}^{\rho=2\cos\phi} \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos\phi \sin\phi \cdot \frac{1}{4} (16 \cos^4\phi) \, d\phi \, d\theta$$

$$= 4 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos^5\phi \sin\phi \, d\phi \, d\theta$$

$$= 4 \int_0^{2\pi} -\frac{1}{6} \cos^6\phi \Big|_{\phi=0}^{\phi=\frac{\pi}{2}} \, d\theta$$

$$= -\frac{2}{3} \int_0^{2\pi} \left(\cos^6\frac{\pi}{2} - \cos^6 0\right) d\theta$$

$$= -\frac{2}{3} \int_0^{2\pi} (0 - 1) \, d\theta$$

$$= \frac{2}{3} \cdot \theta \Big|_0^{2\pi} = \frac{4}{3}\pi \text{ gm.}$$

Find summaries and conversion formulas for all three Coordinate Systems on the next page.

A BRIEF SUMMARY OF MULTIPLE INTEGRATION AND COORDINATE SYSTEMS

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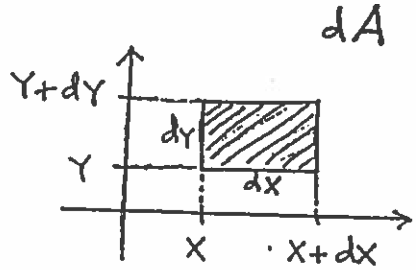
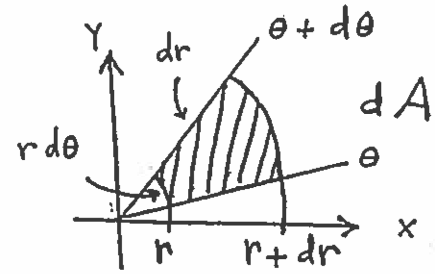


I. Double Integrals

A. Rectangular Coordinates : $dA = dy\,dx$

B. Polar Coordinates : $dA = (r\,d\theta)\,dr = r\,dr\,d\theta$



II. Triple Integrals

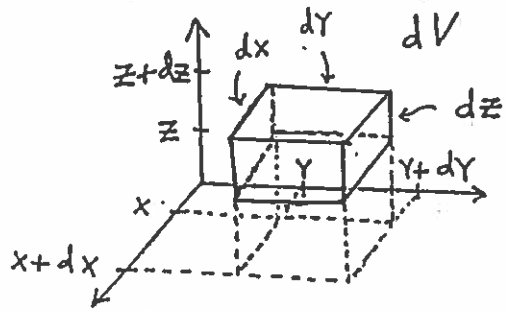
A. Rectangular Coordinates : $dV = dz\,dy\,dx$

B. Cylindrical Coordinates : $dV = dz\,dr\,(r\,d\theta) = r\,dz\,dr\,d\theta$

C. Spherical Coordinates : $dV = d\rho\,(\rho\,d\phi)\,(\rho\,\sin\phi\,d\theta) = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

III. Relationships Between Coordinate Systems

A. Two-Dimensional— rectangular (x, y) , polar (r, θ)

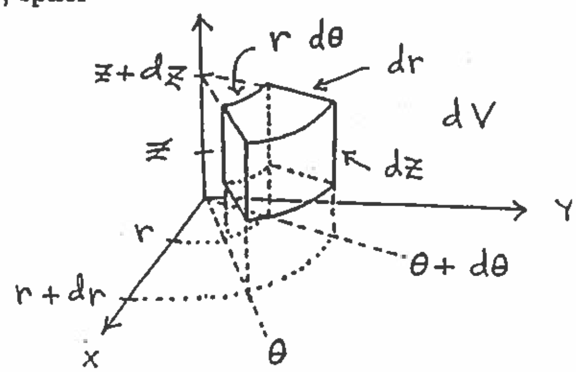


$$x = r\cos\theta \ , \ y = r\sin\theta$$

and

$$r^2 = x^2 + y^2 \ , \ \tan\theta = \frac{y}{x}$$

B. Three-Dimensional— rectangular (x, y, z) , cylindrical (r, θ, z), spherical (ρ, θ, ϕ)



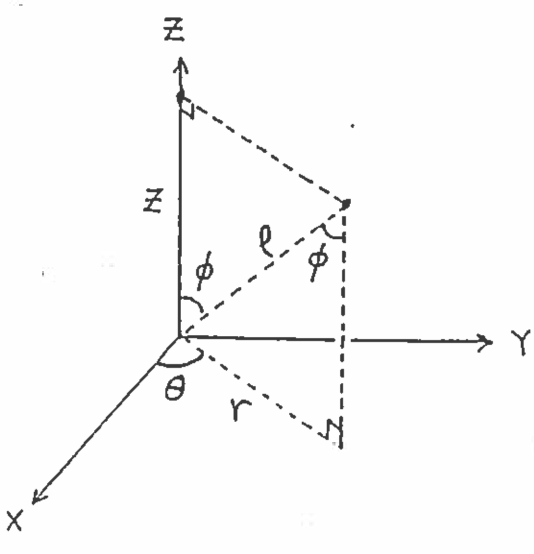
$$r = \rho\ \sin\phi$$

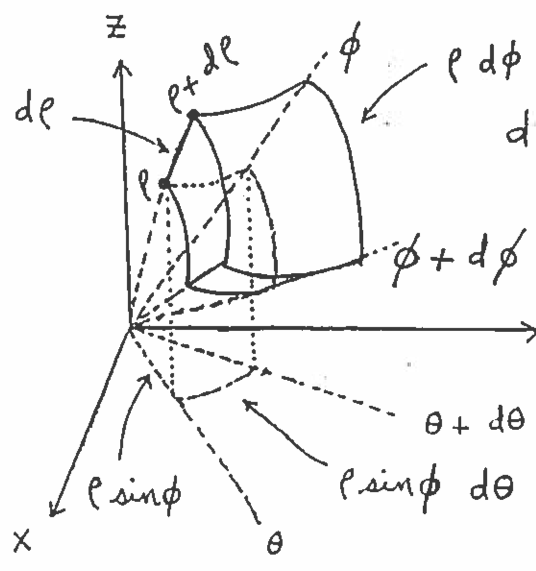
and

$$x = (\rho\ \sin\phi)\ \cos\theta\ = \rho\ \cos\theta\ \sin\phi\ ,$$
$$y = (\rho\ \sin\phi)\ \sin\theta\ = \rho\ \sin\theta\ \sin\phi\ ,$$
$$z = \rho\ \cos\phi$$

and

$$\rho^2 = x^2 + y^2 + z^2$$





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Applications of Triple Integrals

Let R be a solid region in three-dimensional space and let $\delta(P)$ be the density of the region at point $P = (x, y, z)$.

1.) VOLUME : $\displaystyle\int_R 1\,dV$ represents the *volume* of region R.

2.) AVERAGE VALUE : $\displaystyle\frac{1}{Volume\ of\ R}\int_R f(x,y,z)\,dV$ represents the *average value* of function $w = f(x,y,z)$ over region R.

3.) MASS : $\displaystyle\int_R \delta(P)\,dV$ represents the *mass* of region R.

4.) MOMENT :

a.) $\displaystyle\int_R (x-a)\delta(P)\,dV$ represents the *moment* of region R about the plane $x = a$.

b.) $\displaystyle\int_R (y-b)\delta(P)\,dV$ represents the *moment* of region R about the plane $y = b$.

c.) $\displaystyle\int_R (z-c)\delta(P)\,dV$ represents the *moment* of region R about the plane $z = c$.

5.) CENTER OF MASS, $(\bar{x}, \bar{y}, \bar{z})$:

a.) $\bar{x} = \dfrac{\int_R x\delta(P)\,dV}{\int_R \delta(P)\,dV}$ represents the *x-coordinate* of the center of mass of region R.

b.) $\bar{y} = \dfrac{\int_R y\delta(P)\,dV}{\int_R \delta(P)\,dV}$ represents the *y-coordinate* of the center of mass of region R.

c.) $\bar{z} = \dfrac{\int_R z\delta(P)\,dV}{\int_R \delta(P)\,dV}$ represents the *z-coordinate* of the center of mass of region R.

6.) CENTROID, $(\bar{x}, \bar{y}, \bar{z})$:

a.) $\bar{x} = \dfrac{\int_R x\,dV}{\int_R 1\,dV}$ represents the *x-coordinate* of the centroid of region R.

b.) $\bar{y} = \dfrac{\int_R y\,dV}{\int_R 1\,dV}$ represents the *y-coordinate* of the centroid of region R.

c.) $\bar{z} = \dfrac{\int_R z\,dV}{\int_R 1\,dV}$ represents the *z-coordinate* of the center of mass of region R.

NOTE : The formulas for centroid follow immediately from the formulas for center of mass by letting density $\delta(P) = 1$.

7.) MOMENT OF INERTIA : $\displaystyle\int_R (distance)^2 \delta(P)\,dV$ represents the *moment of inertia* of region R, where *distance* refers to the distance from point $P = (x, y, z)$ in region R to either a point or axis (line) of rotation.