Triple Integrals Over Solid Regions \( R \) Using Rectangular Coordinates

Assume that function \( z = f(P) \) is defined on a solid region \( R \) in 3D-space, i.e., \( z = f(x, y, z) \). Partition \( R \) into \( n \) parts \( R_1, R_2, \ldots, R_n \) of volumes \( \Delta V_1, \Delta V_2, \) and \( \Delta V_n \), resp. Pick sampling points \( P_i = (x_i, y_i, z_i) \) in \( R_i \) for \( i = 1, 2, \ldots, n \). Define the diameter of \( R_i \), \( \text{diam}(R_i) \), to be the maximum distance between points in \( R_i \) and define the mesh of the partition to be

\[
\text{mesh} = \max_{1 \leq i \leq n} (\text{diam}(R_i)).
\]

Then

\[
\iiint_R f(P) \, dV = \lim_{\text{mesh} \to 0} \sum_{i=1}^{n} f(P_i) \cdot \Delta V_i.
\]

If \( \Delta V_i = (\Delta z_i) (\Delta y_i) (\Delta x_i) \) for \( i = 1, 2, \ldots, n \), then

\[
\iiint_R f(P) \, dV = \iiint_R f(P) \, dz \, dy \, dx.
\]
1.) $\iiint dV = \text{Volume of } R$

2.) $\iiint \sigma(p) \, dV = \text{Mass of } R$, where $\sigma(p)$ is density \( \left( \frac{\text{mass}}{\text{volume}} \right) \)

3.) The average value of \( z = f(p) \) on solid region \( R \) is

\[
\text{AVE} = \frac{1}{\text{volume of } R} \cdot \iiint f(p) \, dV
\]