Section 13.3
Thomas Calculus
11th Ed.

The Unit Tangent Vector to a Path C in Space

Definition: Let $\vec{r}(t)$ be a vector function which plots a path C in Space (2D or 3D). The Unit Tangent Vector to path C is

$$\vec{T}(t) = \frac{1}{|\vec{v}(t)|} \cdot \vec{v}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}.$$

Note: 1.) $\vec{T}(t)$ is a unit vector (has length 1).

2.) $\vec{T}(t)$ is tangent to path C and points in the direction of motion since $\vec{T}(t)$ is a multiple of the velocity vector $\vec{v}(t)$.

Multiple Derivative Notations:

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

$$= \frac{\frac{d}{dt}\vec{r}(t)}{\frac{ds}{dt}} \quad \left(\text{speed} = \frac{ds}{dt} = |\vec{v}(t)|\right)$$

$$= \frac{d\vec{r}(t)}{dt} \cdot \frac{dt}{ds} \quad$$ (assume time $t = t(s)$ is a function of arc length s.)

$$= \frac{d\vec{r}(t)}{ds}.$$

There are situations where we may want to dicuss vector function

$$\vec{r}(t) = \vec{r}(t(s))$$

as a function of its Arc Length s instead of time t.

Recall From Section 13.1:

The acceleration of motion along path C is given by

$$a(t) = \frac{\vec{v}(t) \cdot \vec{a}(t)}{|\vec{v}(t)|}$$

$$= \frac{\vec{v}(t)}{|\vec{v}(t)|} \cdot \vec{a}(t)$$

$$= \vec{T}(t) \cdot \vec{a}(t),$$

i.e., the acceleration of motion is equal to the Dot Product of the Unit Tangent Vector and the Acceleration Vector.

Example : Consider the vector function $\vec{r}(t) = (t^2)\vec{i} + (e^{t-2})\vec{j}$.

1.) Find the Unit Tangent Vector $\vec{T}(t)$.

2.) Find $\vec{T}(2)$.

3.) Find the acceleration of motion when $t = 2$.

1.) $\vec{r}(t) = (t^2)\vec{i} + (e^{t-2})\vec{j} \xrightarrow{D}$

$\vec{v}(t) = (2t)\vec{i} + (e^{t-2})\vec{j} \longrightarrow$

$|\vec{v}(t)| = \sqrt{(2t)^2 + (e^{t-2})^2}$, so

the Unit Tangent Vector is

$$\vec{T}(t) = \frac{(2t)\vec{i} + (e^{t-2})\vec{j}}{\sqrt{4t^2 + e^{2t-4}}}.$$

2.) $$\vec{T}(2) = \frac{(4)\vec{i} + (e^0)\vec{j}}{\sqrt{16 + e^0}}$$

$$= \frac{4}{\sqrt{17}}\vec{i} + \frac{1}{\sqrt{17}}\vec{j}.$$

3.) $\vec{a}(t) = (2)\vec{i} + (e^{t-2})\vec{j}$, ~~so~~

$\vec{a}(2) = (2)\vec{i} + (e^0)\vec{j} = (2)\vec{i} + (1)\vec{j} \longrightarrow$

~~the acceleration of motion~~ when $t=2$ is

$$a(2) = \vec{T}(2) \bullet \vec{a}(2)$$

$$= \left(\frac{4}{\sqrt{17}}\vec{i} + \frac{1}{\sqrt{17}}\vec{j}\right) \bullet \left(2\vec{i} + \vec{j}\right)$$

$$= \frac{8}{\sqrt{17}} + \frac{1}{\sqrt{17}} = \frac{9}{\sqrt{17}}$$

Example: Consider the vector function

$$\vec{r}(t) = (\ln(t+1))\vec{i} + (t^2)\vec{j} + (t^3 - 2t)\vec{k}$$

1.) Find the Unit Tangent Vector $\vec{T}(t)$.

2.) Find $\vec{T}(0)$.

3.) Find the acceleration of motion when $t = 0$.

1.) $\vec{r}(t) = (\ln(t+1))\vec{i} + (t^2)\vec{j} + (t^3 - 2t)\vec{k} \xrightarrow{D}$

$\vec{v}(t) = \left(\frac{1}{t+1}\right)\vec{i} + (2t)\vec{j} + (3t^2 - 2)\vec{k} \longrightarrow$

$|\vec{v}(t)| = \sqrt{\left(\frac{1}{t+1}\right)^2 + (2t)^2 + (3t^2 - 2)^2}$, so

the Unit Tangent Vector is

$$\vec{T}(t) = \frac{\left(\frac{1}{t+1}\right)\vec{i} + (2t)\vec{j} + (3t^2 - 2)\vec{k}}{\sqrt{\left(\frac{1}{t+1}\right)^2 + 4t^2 + (3t^2 - 2)^2}} .$$

2.) $$\vec{T}(0) = \frac{(1)\vec{i} + (0)\vec{j} + (-2)\vec{k}}{\sqrt{(1)^2 + (0) + (-2)^2}}$$

$$= \frac{1}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}\vec{k} .$$

3.) $\vec{a}(t) = \left(\frac{-1}{(t+1)^2}\right)\vec{i} + (2)\vec{j} + (6t^2)\vec{k} \rightarrow$

the acceleration of motion when $t=0$ is

$$a(0) = \vec{T}(0) \cdot \vec{a}(0)$$

$$= \left(\frac{1}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}\vec{k}\right) \cdot \left((-1)\vec{i} + (2)\vec{j}\right)$$

$$= \frac{-1}{\sqrt{5}}$$