Example : Consider the vector function

$$\vec{r}(t) = \sin(t^2) \cdot \vec{i} + \cos(t^2) \cdot \vec{j}$$

and its path C in 2D-Space.

1.) Find the arc Length along C for $t=0$ to t.

2.) Find the arc Length along C for $t=0$ to $t=2\pi$.

3.) a.) Write time t as a function of arc Length s.

b.) Write the vector function $\vec{r}(t)$ as a function of s.

c.) Find the derivative

$\frac{d\vec{r}}{ds}$, the derivative of $\vec{r}$ with respect to arc length s.

i.) Find $\frac{d\vec{r}}{ds}$ for $s = 0$.

ii.) Find $\frac{d\vec{r}}{ds}$ for $s = \frac{\pi}{2}$.

1.) $\vec{r}(t) = \sin(t^2)\cdot\vec{i} + \cos(t^2)\cdot\vec{j} \xrightarrow{D}$

$\vec{v}(t) = 2t\cos(t^2)\cdot\vec{i} - 2t\sin(t^2)\vec{j}$, so arc length is

$$s = \int_0^t |\vec{v}(\tau)|\, d\tau$$

$$= \int_0^t \sqrt{(2\tau\cos(\tau^2))^2 + (-2\tau\sin(\tau^2))^2}\cdot d\tau$$

$$= \int_0^t \sqrt{4\tau^2\cos^2(\tau^2) + 4\tau^2\sin^2(\tau^2)}\, d\tau$$

$$= \int_0^t \sqrt{4\tau^2(\cos^2(\tau^2)+\sin^2(\tau^2))}\, d\tau$$

$$= \int_0^t \sqrt{4\tau^2(1)}\, d\tau$$

$$= \int_0^t 2\tau\, d\tau = \tau^2\Big|_0^t = t^2, \text{ i.e.,}$$

arc length $\boxed{s=t^2}$.

2.) If $t=2\pi$, then arc length

$$s=(2\pi)^2 = 4\pi^2.$$

3.) a.) If $s=t^2$, then $\boxed{t=\sqrt{s}}$.

b.) $\vec{r}(t) = \vec{r}(t(s))$

$$= (\sin s)\,\vec{i} + (\cos s)\,\vec{j}$$

c.) $\dfrac{d\vec{r}}{ds} = (\cos s)\,\vec{i} + (-\sin s)\,\vec{j}$

i.) If $s=0$, then

$$\frac{d\vec{r}}{ds} = (\cos 0)\vec{i} + (-\sin 0)\vec{j}$$
$$= (1)\vec{i} + (0)\vec{j}$$
$$= \vec{i} \ .$$

ii.) If $s=\frac{\pi}{2}$, then

$$\frac{d\vec{r}}{ds} = \left(\cos\frac{\pi}{2}\right)\vec{i} + \left(-\sin\frac{\pi}{2}\right)\vec{j}$$
$$= (0)\vec{i} + (-1)\vec{j}$$
$$= -\vec{j}$$

<u>NOTE</u>: We will learn later that

$$\frac{d\vec{r}}{ds} = \vec{T}(t) \ ,$$

the <u>Unit</u> <u>Tangent</u> <u>Vector</u>.