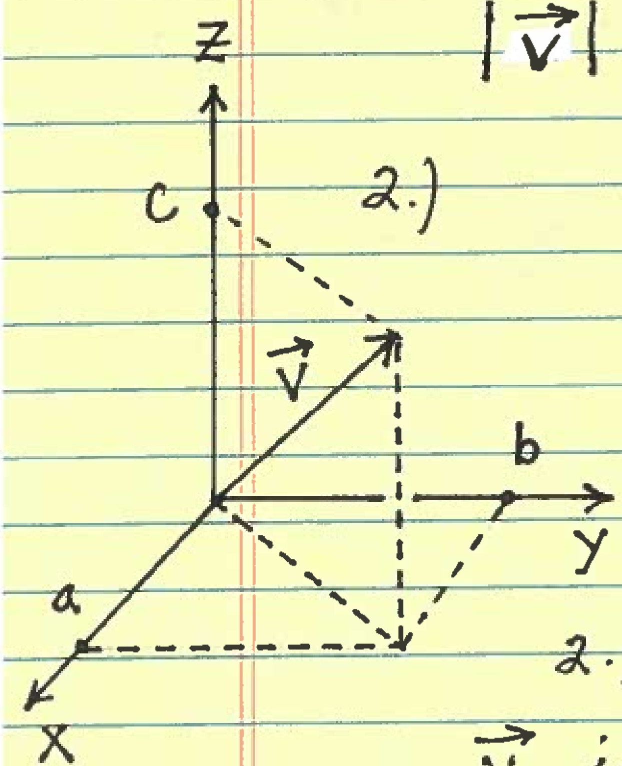
Section 13.1
Thomas Calculus
11th Ed.

Vectors Review

Recall: I.) Vector $\boxed{\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}}$ in 3D-Space is a quantity with DIRECTION and MAGNITUDE (length), where $\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$, and $\vec{k} = \overrightarrow{(0,0,0)}$.

1.) The Magnitude of $\vec{v}$ ~~is~~

$$|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$$



2.)

(Recall: a vector $\vec{u}$ is called a unit vector if its magnitude is 1.)

2.) The Direction of $\vec{v}$ is the unit vector

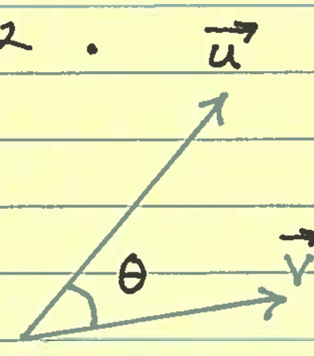
$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \frac{a}{\sqrt{a^2+b^2+c^2}}\vec{i}$$

$$+ \frac{b}{\sqrt{a^2+b^2+c^2}}\vec{j} + \frac{c}{\sqrt{a^2+b^2+c^2}}\vec{k}$$

NOTE: For vectors in 2D-Space assume that $c = 0$.

II.) If $\vec{u} = a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$ and $\vec{v} = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}$, then the DOT PRODUCT of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = a_1a_2 + b_1b_2 + c_1c_2 .$$



1.) The angle θ between vectors $\vec{u}$ and $\vec{v}$ is given by (derived by using the Law of Cosines) a.) $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$

b.) $\vec{u}$ is <u>orthogonal</u> to $\vec{v}$, $\vec{u} \perp \vec{v}$, if $\vec{u} \cdot \vec{v} = 0$.

2.) The magnitude of vector $\vec{w} = a\vec{i} + b\vec{j} + c\vec{k}$ is $|\vec{w}| = \sqrt{a^2+b^2+c^2}$, and

$$|\vec{w}|^2 = a^2+b^2+c^2 = \vec{w} \cdot \vec{w}.$$

III.) Line L passing through the point (a, b, c) and in the direction of vector $\vec{u} = u_1\vec{i} + v_1\vec{j} + w_1\vec{k}$ is given parametrically by

$$L: \begin{cases} x = a + u_1 t \\ y = b + v_1 t \quad \text{for } -\infty < t < \infty \\ z = c + w_1 t \end{cases}$$

Vector Functions

Definition: The vector function

$$\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

assigns a vector in 3D-Space to each real number t (time) in its domain.

Example: (2D-Space) Let

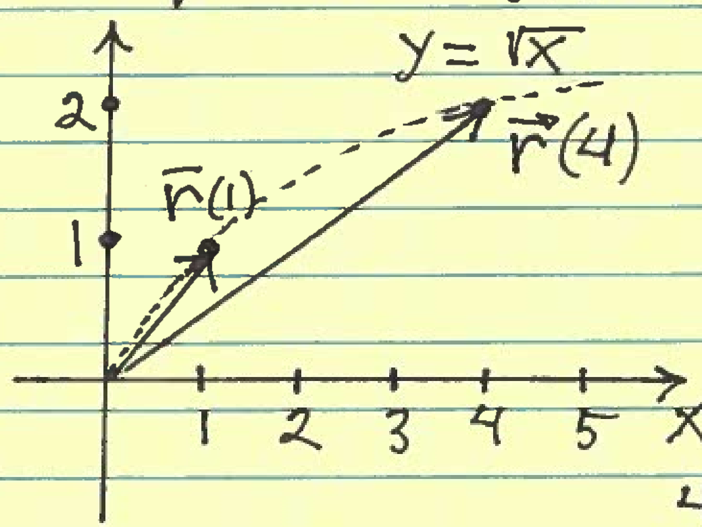
$\vec{r}(t) = t\,\vec{i} + \sqrt{t}\,\vec{j}$ for $t \geq 0$; then

$\begin{cases} x = t \\ y = \sqrt{t} \end{cases} \rightarrow y = \sqrt{x}$;

and

$\vec{r}(1) = 1 \cdot \vec{i} + 1 \cdot \vec{j}$,

$\vec{r}(4) = 4 \cdot \vec{i} + 2 \cdot \vec{j}$

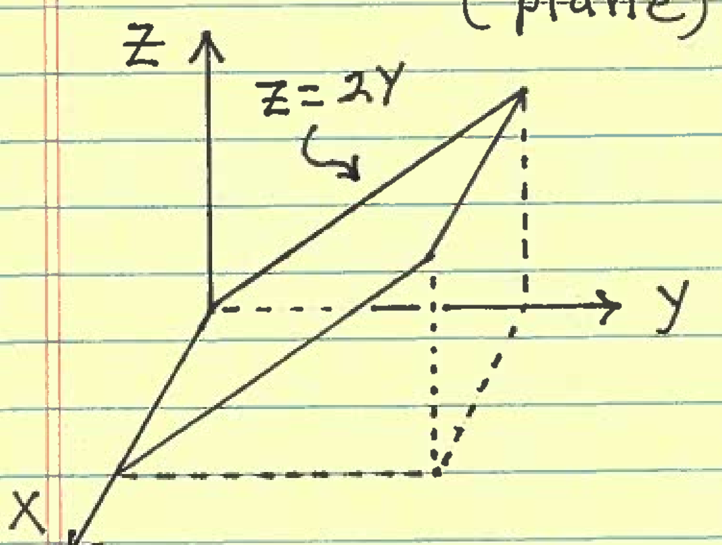


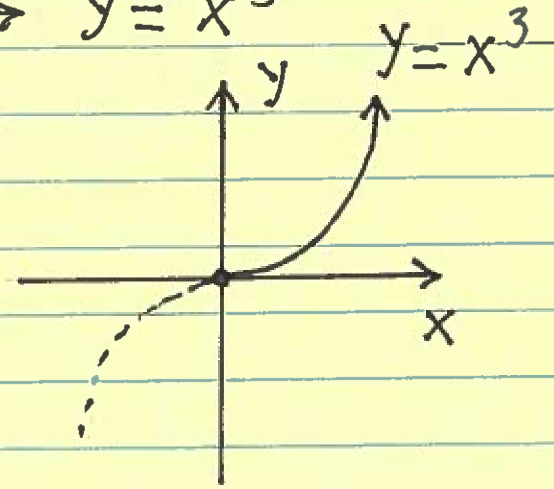
Note: If we assume that the tail of each vector is at the origin, then the tips of each vector $\vec{r}(t)$ trace out a path C in space.

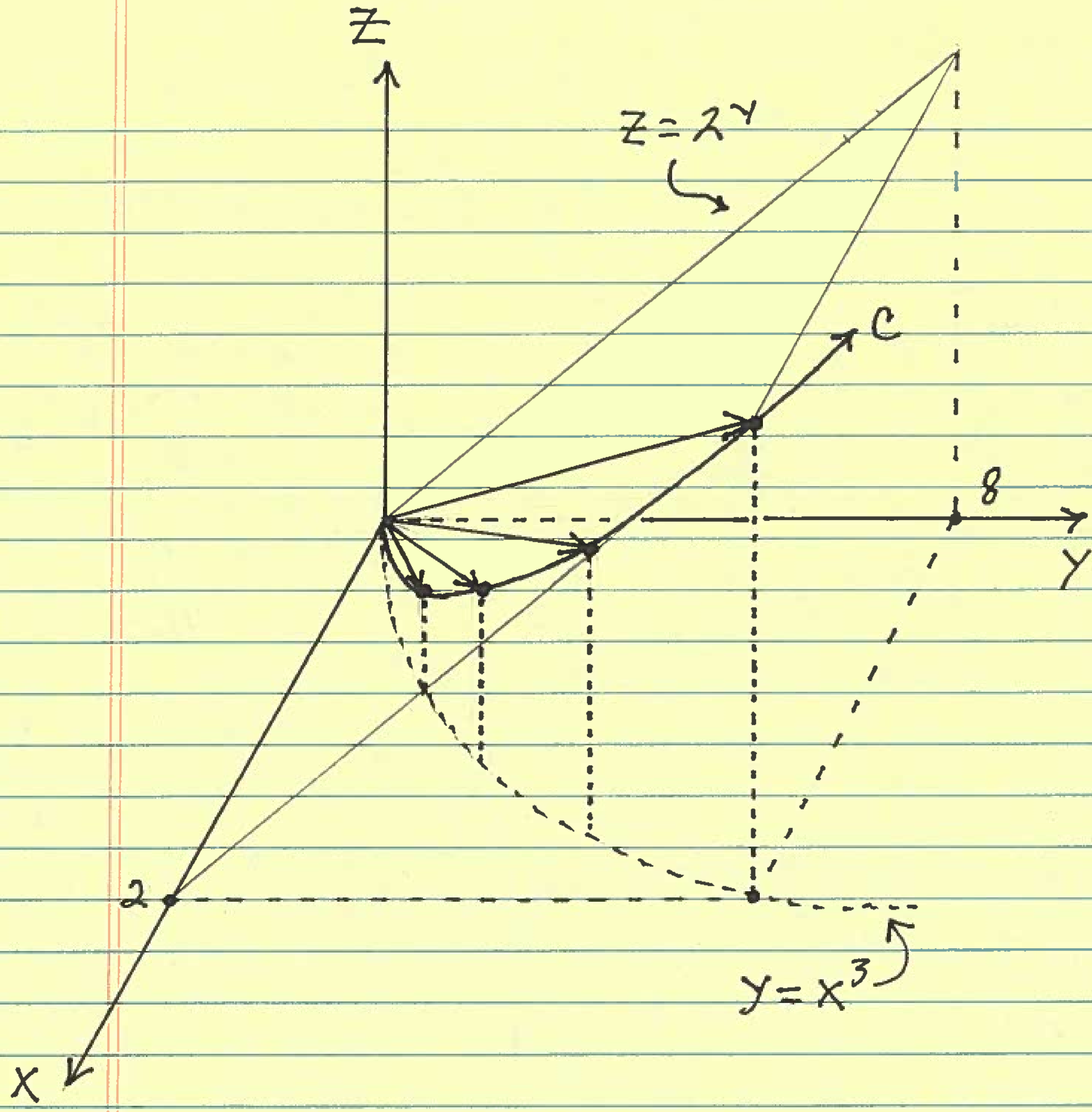
Example: (3D-Space) Plot the path C determined by the vector function

$$\vec{r}(t) = \sqrt[3]{t}\,\vec{i} + t\,\vec{j} + 2t\,\vec{k} \quad \text{for } t \geq 0.$$

$$\begin{cases} x = \sqrt[3]{t} \\ y = t \\ z = 2t \end{cases} \qquad \begin{array}{l} \nearrow\; x = \sqrt[3]{y} \rightarrow y = x^3 \\ \rightarrow\; z = 2y \text{ (plane)} \end{array}$$

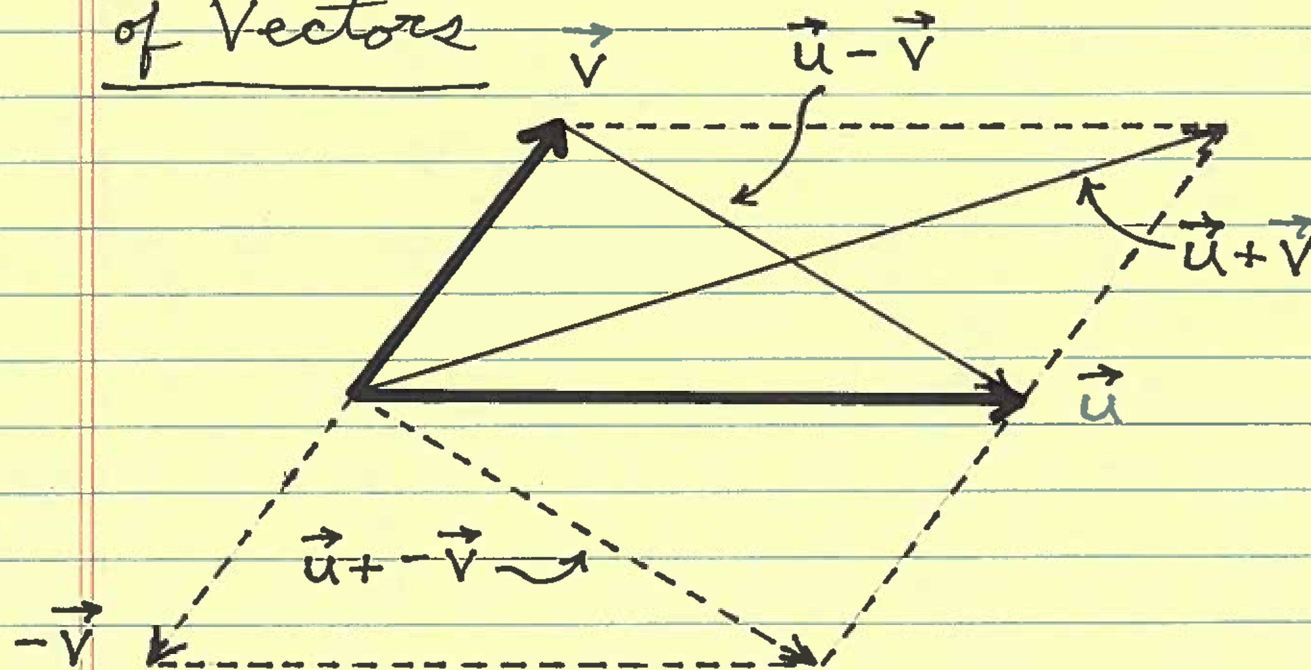






Z
$z = 2^y$
C
8
Y
2
$y = x^3$
X

Recall: **Addition and Subtraction of Vectors**



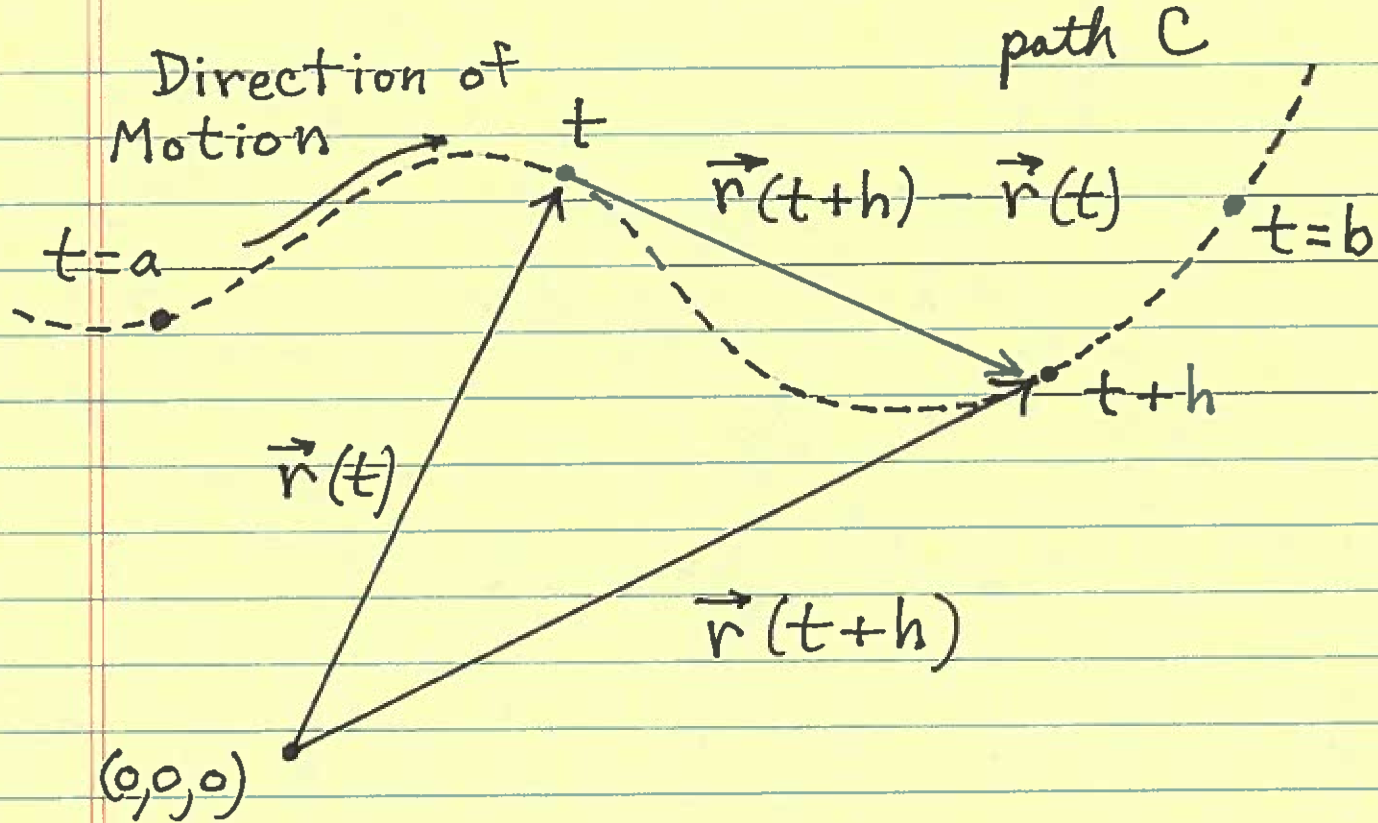
Definition: The **derivative** of vector function

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

is

$$\frac{d}{dt}\vec{r}(t) = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k} .$$



Note that the magnitude of $\vec{r}\,'(t)$ is

$$\left|\vec{r}\,'(t)\right| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\ ,$$

and from Math 21C we know ~~that arc length~~ s from t=a to t is

$$s = \int_a^t \sqrt{(f'(\tau))^2 + (g'(\tau))^2 + (h'(\tau))^2}\, d\tau\ ;$$

thus the speed of motion at

time t along path C is (FTC 1)

$$\frac{ds}{dt} = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} = |\vec{r}\,'(t)|.$$

SUMMARY:

1.) Position Vector $\left.\right\}$ $\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$

2.) Velocity Vector $\left.\right\}$ $\vec{v}(t) = \vec{r}\,'(t) = f'(t)\,\vec{i} + g'(t)\,\vec{j} + h'(t)\,\vec{k}$

a.) $\vec{v}(t)$ points in the Direction of Motion.

b.) $\vec{v}(t)$ is TANGENT to path C.

c.) The magnitude of $\vec{v}(t)$, $|\vec{v}(t)|$, is the speed of motion along path C.

3.) <u>Acceleration Vector</u>:

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$$

$$= f''(t)\,\vec{i} + g''(t)\,\vec{j} + h''(t)\,\vec{k}$$

<u>Differentiation Rules for Vector Functions</u>: assume that $\vec{u}(t)$ and $\vec{v}(t)$ are vector functions, $\vec{C} = a\vec{i} + b\vec{j} + c\vec{k}$ is a constant vector, c is a constant, and $y = f(t)$ is a real-valued function

1.) $\frac{d}{dt}\vec{C} = \vec{0} = (0)\vec{i} + (0)\vec{j} + (0)\vec{k}$

2.) $\frac{d}{dt}\left(c\,\vec{u}(t)\right) = c \cdot \frac{d}{dt}\vec{u}(t) = c\,\vec{u}'(t).$

3.) $\frac{d}{dt}\left(\vec{u}(t) \pm \vec{v}(t)\right) = \vec{u}'(t) \pm \vec{v}'(t)$

4.) (DOT PRODUCT RULE)

$$\frac{d}{dt}\left(\vec{u}(t) \cdot \vec{v}(t)\right) = \vec{u}(t) \cdot \vec{v}'(t) + \vec{u}'(t) \cdot \vec{v}(t)$$

5.) (CROSS PRODUCT RULE)

$$\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t))=\vec{u}(t)\times\vec{v}'(t)+\vec{u}'(t)\times\vec{v}(t)$$

6.) (CHAIN RULE)

$$\frac{d}{dt}\,\vec{u}(f(t))=\vec{u}'(f(t))\cdot f'(t)$$

<u>Example</u>: assume that a vector function $\vec{r}(t)$ has constant length, i.e.,

$$|\vec{r}(t)|=c\text{, a constant.}$$

Let's show that the velocity vector $\vec{v}(t)=\vec{r}'(t)$ is orthogonal to the position vector $\vec{r}(t)$, i.e.,

$$\vec{r}(t)\perp\vec{r}'(t)\text{, i.e., }\vec{r}(t)\cdot\vec{r}'(t)=0.$$

Start $|\vec{r}(t)|=c \longrightarrow$

$$\left|\vec{r}(t)\right|^2 = c^2 \longrightarrow$$

$$\vec{r}(t) \cdot \vec{r}(t) = c^2 \xrightarrow{D}$$

$$\vec{r}(t) \cdot \vec{r}\,'(t) + \vec{r}\,'(t) \cdot \vec{r}(t) = 0 \longrightarrow$$

$$2\,\vec{r}(t) \cdot \vec{r}\,'(t) = 0 \longrightarrow$$

$$\vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

Q.E.D.

Example: Consider the vector function given by

$$\vec{r}(t)=\left(\tfrac{1}{2}t^2\right)\vec{i}+(t)\vec{j}+\left(\tfrac{1}{3}t^3\right)\vec{k}$$

for $t \geq 0$, and its path C in 3D-Space. Find the line L in parametric form, which is tangent to path C when $t = 2$:

$$\vec{r}(t)=\left(\tfrac{1}{2}t^2\right)\vec{i}+(t)\vec{j}+\left(\tfrac{1}{3}t^3\right)\vec{k} \xrightarrow{D}$$

$$\vec{v}(t) = (t)\vec{i}+(1)\vec{j}+(t^2)\vec{k} \; ;$$

the point at $t=2$ is

$$\vec{r}(2)=(2)\vec{i}+(2)\vec{j}+\left(\tfrac{8}{3}\right)\vec{k} \longrightarrow$$

$\left(2, 2, \frac{8}{3}\right)$; the velocity vector is tangent to path C at $t=2$:

$$\vec{v}(2)=(2)\vec{i}+(1)\vec{j}+(4)\vec{k}, \text{ so}$$

tangent line L is

$$L : \begin{cases} x = 2 + 2t \\ y = 2 + t \\ z = \frac{8}{3} + 4t \end{cases}$$