Section 16.2
Thomas Calculus
11th Ed.

Vector Fields and Work Along a Path C

Ex: Assume that water flows through a space and at each point $P = (x, y, z)$ we know the speed and direction of water flow:

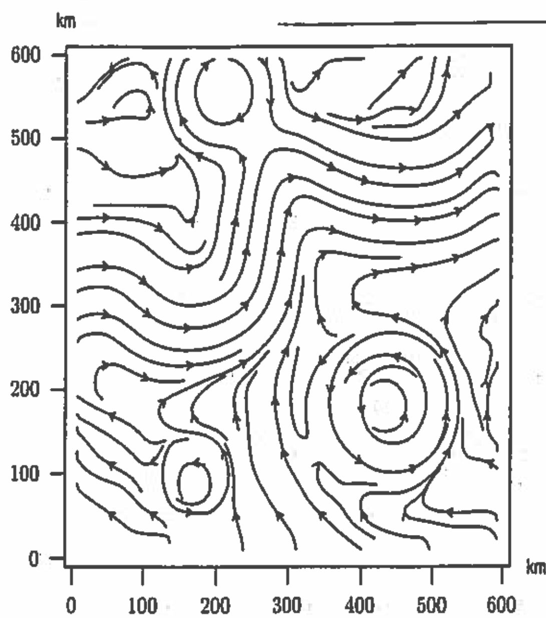


Figure 17.32: Flow lines for objects in the Gulf stream with different starting points

(Calculus by Hughes-Hallett, 3rd. edition, p. 805)

We call this representation a vector field.

Def: A vector field is a function which

assigns a vector to each point in space (also called force field, velocity field, gradient field, gravitational field, etc.)

1.) (2D-space)

$$\vec{F}(x,y) = M(x,y)\cdot\vec{i} + N(x,y)\vec{j}$$

2.) (3D-space)

$$\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$$

Ex: The following are vector fields.

1.) $\vec{F}(x,y) = (xy)\vec{i} + (x^2+y^2)\vec{j}$

2.) $\vec{F}(x,y,z) = (\cos x)\vec{i} + (\sin y)\vec{j} + \tan(xy)\cdot\vec{k}$

Def: The gradient field of scalar function $f(x,y,z)$ is the vector field

$$\boxed{\vec{\nabla} f = f_x\cdot\vec{i} + f_y\cdot\vec{j} + f_z\cdot\vec{k}}$$

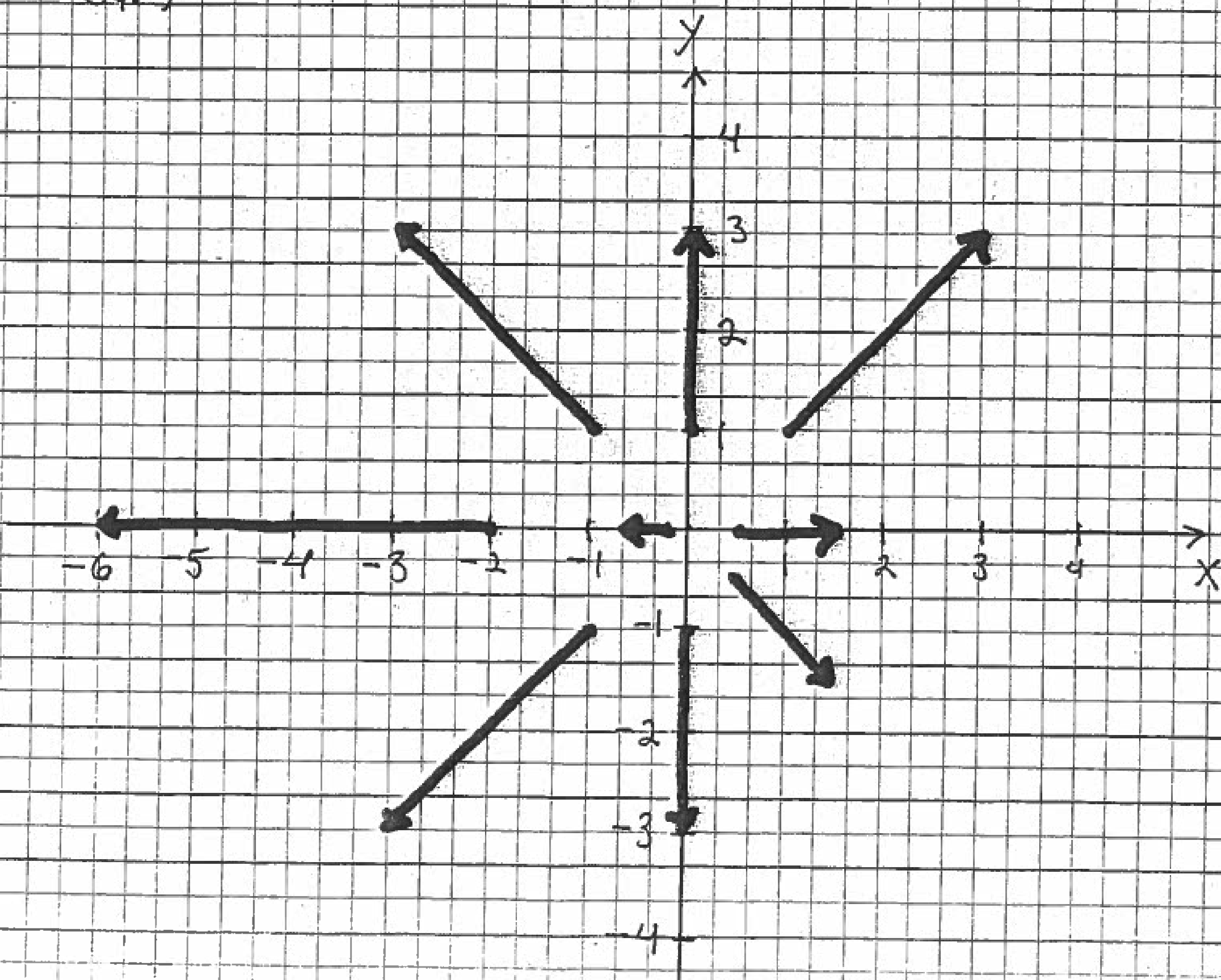
Ex: Determine the gradient field for $f(x,y) = x^2+y^2$ and plot gradient vectors for the following points:

$(\frac{1}{2},0)$, $(1,1)$, $(0,1)$, $(-1,1)$, $(\frac{-1}{4},0)$, $(-2,0)$, $(-1,-1)$, $(0,-1)$, and $(\frac{1}{2},\frac{-1}{2})$

$\vec{\nabla} f = f_x \cdot \vec{i} + f_y \cdot \vec{j}$ $\longrightarrow$

$\vec{\nabla} f(x,y) = (2x)\vec{i} + (2y)\vec{j}$;

(x,y)	$\vec{\nabla} f(x,y)$	(x,y)	$\vec{\nabla} f(x,y)$
$(\frac{1}{2}, 0)$	$\vec{i}$	$(-2, 0)$	$-4\vec{i}$
$(1,1)$	$2\vec{i} + 2\vec{j}$	$(-1,-1)$	$-2\vec{i} - 2\vec{j}$
$(0,1)$	$2\vec{j}$	$(0,-1)$	$-2\vec{j}$
$(-1,1)$	$-2\vec{i} + 2\vec{j}$	$(\frac{1}{2}, -\frac{1}{2})$	$\vec{i} - \vec{j}$
$(-\frac{1}{4}, 0)$	$-\frac{1}{2}\vec{i}$		

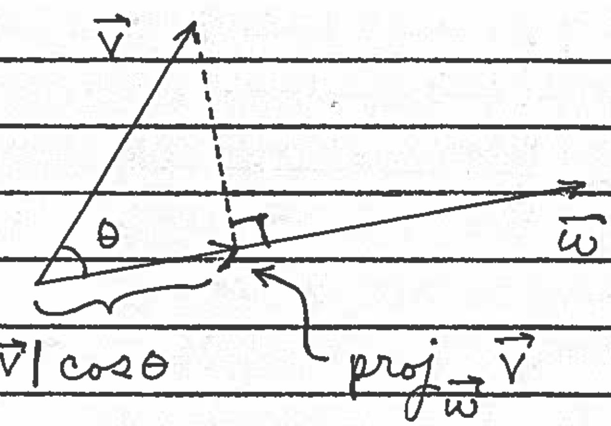


Recall: Let $z = f(x,y)$ be a surface in 3D-space. Then the gradient vector

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}$$

1.) is $\perp$ to the level curve $f(x,y) = c$.
2.) points in the direction of the maximum directional derivative.
3.) has magnitude equal to the value of the maximum directional derivative.

Recall: Let $\vec{v}$ and $\vec{w}$ be vectors. The projection of $\vec{v}$ onto $\vec{w}$ is



$$\text{proj}_{\vec{w}}\vec{v} = |\vec{v}|\cos\theta \cdot \frac{\vec{w}}{|\vec{w}|}$$

$$= \cancel{|\vec{v}|}\left(\frac{\vec{v}\cdot\vec{w}}{\cancel{|\vec{v}|}\,|\vec{w}|}\right)\frac{\vec{w}}{|\vec{w}|}$$

$$= \left(\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\right)\vec{w}, \text{ i.e.,}$$

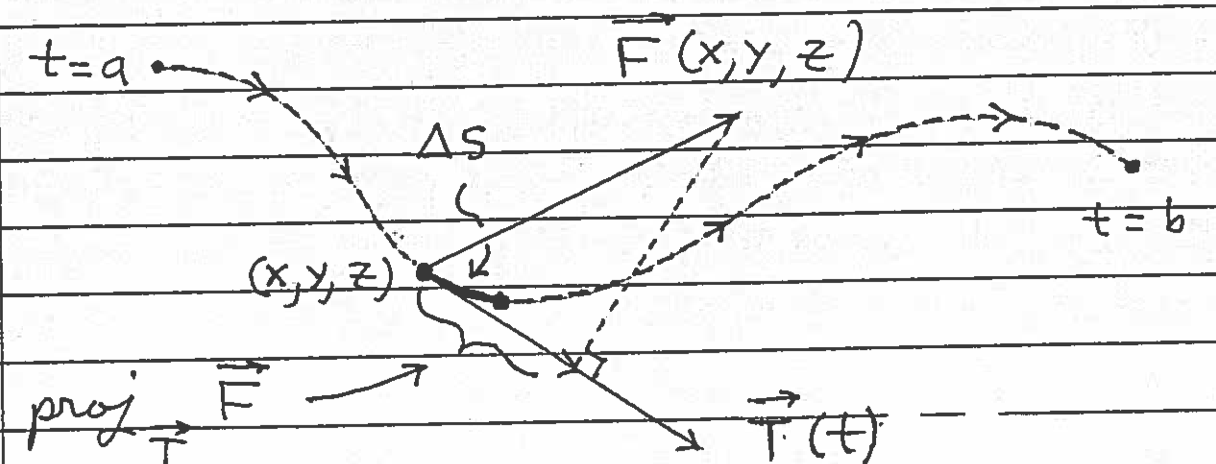
$$\text{proj}_{\vec{w}}\vec{v} = \left(\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\right)\vec{w}$$; if $\vec{w}$ is a unit vector, then $|\vec{w}| = 1$ and $\boxed{\text{proj}_{\vec{w}}\vec{v} = (\vec{v}\cdot\vec{w})\,\vec{w}}$.

We call $(\vec{v} \cdot \vec{w})$ the scalar component of $\vec{v}$ in the direction of $\vec{w}$.

Def: Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ be a vector field of force vectors (force field). Let $C: \vec{r}(t)$ for $a \le t \le b$ be a curve in space on which $\vec{F}$ is defined. We want a formula for the work required by this force field to push a particle along path C.

(Recall: Work = Force × Distance)

Let $\vec{T}(t)$ be the unit tangent vector to C. Then $\text{proj}_{\vec{T}} \vec{F}$



has scalar component $\vec{F} \cdot \vec{T}$, so that the work done by $\vec{F}$ along path C is

$$\boxed{\text{Work} = \int_C \vec{F} \cdot \vec{T} \, ds}.$$

Method of Computation:

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{T}\,\frac{ds}{dt}\,dt$$

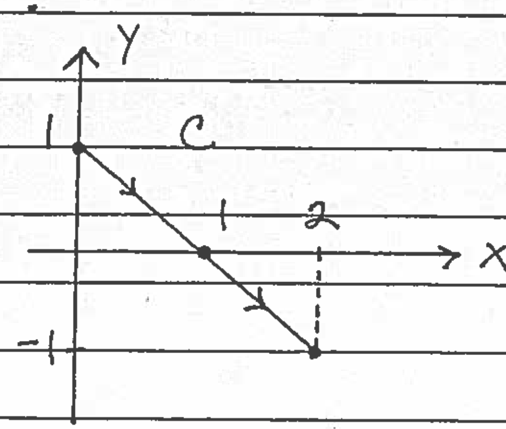
$$= \int_C \vec{F}\cdot\frac{\vec{v}(t)}{\cancel{|\vec{v}(t)|}}\,\cancel{\frac{ds}{dt}}\,dt \qquad \left(|\vec{v}(t)| = \frac{ds}{dt}\right)$$

$$= \int_C \vec{F}\cdot\vec{v}(t)\,dt\,, \text{ i.e.,}$$

$$\boxed{\text{Work} = \int_a^b \vec{F}\cdot\vec{r}\,'(t)\,dt} \qquad \left(\vec{v}(t) = \vec{r}\,'(t)\right)$$

Ex: Compute the work done by the force field $\vec{F}(x,y) = (2x)\,\vec{i} + (2y)\,\vec{j}$ along path C determined by $y = 1 - x$ for $0 \le x \le 2$.

$$C: \begin{cases} x = t \\ y = 1 - t \end{cases} \text{ for } 0 \le t \le 2$$



$$\vec{r}(t) = (t)\,\vec{i} + (1-t)\,\vec{j}$$

$$\vec{r}\,'(t) = (1)\,\vec{i} + (-1)\,\vec{j}\,;$$

$$\vec{F}(x,y) = \vec{F}(x(t), y(t)) = (2t)\,\vec{i} + (2-2t)\,\vec{j}\,;$$

then

$$\text{Work} = \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_C \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^2 \left[(2t)(1) + (2-2t)(-1)\right] dt$$

$$= \int_0^2 (4t-2)\, dt$$

$$= (2t^2 - 2t)\Big|_0^2 = 8 - 4 = 4$$

The next example will include units.

Example: Compute the work done by the force field

$$\vec{F}(x,y,z) = (x)\vec{i} + (y^2)\vec{j} + (2z)\vec{k},$$

where the units of force on each component vector are Newtons, along path C determined by the vector function

$$\vec{r}(t) = (t^2)\vec{i} + (t)\vec{j} + (3t)\vec{k}$$

for $t=0$ to $t=3$, where distance units are meters.

$$C: \vec{r}(t) = (t^2)\vec{i} + (t)\vec{j} + (3t)\vec{k} \xrightarrow{D}$$

$$\vec{v}(t) = (2t)\vec{i} + (1)\vec{j} + (3)\vec{k}, \text{ and}$$

the force field along path C becomes

$$\vec{F}(x,y,z) = (t^2)\vec{i} + (t^2)\vec{j} + (6t)\vec{k}; \text{ then}$$

work done by $\vec{F}$ along path C is

$$\text{Work} = \int_C \vec{F} \cdot \vec{v}\, dt$$

$$= \int_0^3 \left[(t^2)(2t) + (t^2)(1) + (6t)(3) \right] dt$$

$$= \int_0^3 \left[2t^3 + t^2 + 18t \right] dt$$

$$= \left[\frac{1}{2}t^4 + \frac{1}{3}t^3 + 9t^2 \right] \Big|_0^3$$

$$= \frac{81}{2} + 9 + 81$$

$$= \frac{81}{2} + \frac{180}{2} = \frac{261}{2} \text{ joules}$$

Here is Various Notation for Work Integral

Assume that vector field

$\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$

and path C is given by

$\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,k$ for $a \le t \le b$:

$$\text{Work} = \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_a^b \left(M \frac{dg}{dt} + N \frac{dh}{dt} + P \frac{dk}{dt} \right) dt$$

$$= \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$$

$$= \int_a^b \left(M\, dx + N\, dy + P\, dz \right)$$