Math 22A
Kouba
Diagonal, Triangular, and Symmetric Matrices

DEFINITION: A square matrix A is called **DIAGONAL** if all entries **above** and **below** the main diagonal are zero.

DEFINITION: A square matrix A is called **UPPER TRIANGULAR** if all entries **below** the main diagonal are zero.

DEFINITION: A square matrix A is called **LOWER TRIANGULAR** if all entries **above** the main diagonal are zero.

DEFINITION: A square matrix A is called **SYMMETRIC** if $A^T = A$.

RECALL: (Properties of the Transpose of a Matrix)

1.) $(A^T)^T = A$
2.) $(A \pm B)^T = A^T \pm B^T$
3.) $(AB)^T = B^T A^T$
4.) $(kA)^T = k(A^T)$

THEOREM T: Assume matrices A and B are symmetric and the same size. Let k be any constant. Then

1.) A^T is symmetric.
2.) $A + B$ and $A - B$ are symmetric.
3.) kA is symmetric.
4.) AB is symmetric iff $AB = BA$.

PROOF: Assume that matrix A is symmetric, i.e., assume that $A^T = A$. Assume that matrix B is symmetric, i.e., assume that $B^T = B$.

1.) (YOU)
2.) (YOU)
3.) (YOU)
4.) ($\Longrightarrow$) Assume that AB is symmetric, i.e., $(AB)^T = AB$. Show that $AB = BA$. Then

$(AB)^T = AB$ is given. But $(AB)^T = B^T A^T$ (by properties of transpose) $= BA$ (since A and B are symmetric). Thus, $AB = BA$.

($\Longleftarrow$) Assume that $AB = BA$. Show that AB is symmetric, i.e., show that $(AB)^T = AB$. Then

$(AB)^T = B^T A^T$ (by properties of transpose) $= BA$ (since A and B are symmetric) $= AB$ (by assumption). Thus, AB is symmetric. QED

FACTS ABOUT THESE SPECIALTY MATRICES:

I.) Sums, differences, and products of diagonal matrices are diagonal.

II.) Sums, differences, and products of upper triangular matrices are upper triangular.

III.) Sums, differences, and products of lower triangular matrices are lower triangular.

IV.) All elementary matrics which do NOT switch rows are either upper triangular or lower triangular.

V.) A diagonal matrix is invertible iff ALL of the entries on the main diagonal are NONZERO.

THEOREM M: If A is an invertible symmetric matrix, then A^{-1} is symmetric.

PROOF: (ME)

THEOREM Y: If A is an invertible matrix, then AA^T and A^TA are also invertible.

PROOF: (YOU)

EXAMPLE: Let $A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$. What is A^{10} ?

EXAMPLE: Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1/2 \end{pmatrix}$. What is A^{-15} ?

EXAMPLE: Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. What is A^{100} ?

TRUE or FALSE: If A is a symmetric matrix, then A^3 is symmetric.

TRUE or FALSE: If $A^2 - 2A = A^3 - I$, then matrix A is invertible.