Lecture 18

Math 22A
Sec. 4.5
HW #18

Dimension of a Vector Space

Theorem: all bases for a finite-dimensional vector space have the same # of vectors.

proof: (Special case)
Suppose a vector space V has two bases:

$$B_1 = \{\vec{v}_1, \vec{v}_2\}, \quad B_2 = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\},$$

so both sets span V and are linearly independent.
Thus, all vectors in B_2 can be written as linear

combinations of vectors in B_1. It follows that

$$\alpha_1 \vec{w}_1 + \alpha_2 \vec{w}_2 + \alpha \vec{w}_3 = \vec{0} \Rightarrow$$

$$\alpha_1 (a\vec{v}_1 + b\vec{v}_2) + \alpha_2 (c\vec{v}_1 + d\vec{v}_2)$$
$$+ \alpha_3 (e\vec{v}_1 + f\vec{v}_2) = \vec{0} \Rightarrow$$

$$(\alpha_1 a + \alpha_2 c + \alpha_3 e)\vec{v}_1$$
$$+ (\alpha_1 b + \alpha_2 d + \alpha_3 f)\vec{v}_2 = \vec{0} \Rightarrow$$

$$\begin{cases} \alpha_1 a + \alpha_2 c + \alpha_3 e = 0 \\ \alpha_1 b + \alpha_2 d + \alpha_3 f = 0 \end{cases}$$ (by lin. ind. of B_1)

(Homogeneous System of 2 equations, 3 unknowns $\alpha_1, \alpha_2, \alpha_3$) $\Rightarrow$

at least $3-2=1$ free variable $\Rightarrow$ NONZERO SOLUTIONS. This contradicts the lin. ind. of set B_2. Thus, the # of basis elements cannot be different.

Q.E.D.

Theorem: Let V be an n-dimensional vector space, and let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be any basis

a.) If a set S in V has more than n vectors, then S is linearly dependent.

b.) If a set in V has fewer than n vectors, then S does not span V.

proof: Similar ~~to~~ previous theorem.

Def: The <u>dimension</u> of a vector space V is the # of elements in a basis for V. <u>NOTE</u>: we say the zero vector space has dimension zero.

<u>Ex</u>: 1.) $\dim(\mathbb{R}^n) = n$

2.) $\dim(M_{32}) = 6$

3.) $\dim(P_3) = 4$

Plus/Minus Theorem:

Let S be a nonempty set of vectors in a vector space V.

a.) Let S be linearly independent and consider the subspace span(S). If $\vec{v} \in V$, but $\vec{v} \notin$ span(S), then

$S \cup \{\vec{v}\}$ is linearly independent.

b.) assume $\vec{v} \in S$ can be written as a linear combination of other vectors in S. Then

S and $S - \{v\}$ span the same subspace.

~~proof~~: a.) (Special case)

Let $S = \{\vec{v}_1, \vec{v}_2\}$ be linearly independent. Let $\vec{v}_3 \in V$, but $\vec{v}_3 \notin \text{span}(S)$. Show $S \cup \{\vec{v}_3\} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

But $\vec{v}_3 \notin \text{span}(S) \Rightarrow$

(#) $\vec{v}_3 \neq k_1\vec{v}_1 + k_2\vec{v}_2$ for all constants k_1, k_2.

Now let

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0} \Rightarrow$$

(*) $$k_3\vec{v}_3 = -k_1\vec{v}_1 + k_2\vec{v}_2 .$$

case 1: If $k_3 = 0$, then (*) becomes

$$0\vec{v_3} = \vec{0} = (-k_1)v_1 + (-k_2)\vec{v_2} \Rightarrow$$

$$-k_1 = 0 \text{ and } -k_2 = 0 \Rightarrow$$

$k_1 = 0$ and k_2 (by lin. ind of set S)

$\Rightarrow \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is lin. ind.

case 2: If $k_3 \neq 0$, then (*) becomes

$$\vec{v_3} = \left(-\frac{k_1}{k_3}\right)\vec{v_1} + \left(-\frac{k_2}{k_3}\right)\vec{v_2}$$

a contradiction of (#).

Q.E.D.

b.) (Special case)

assume $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

and $\vec{v}_3 = a\vec{v}_1 + b\vec{v}_2$, then

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$$

$$= \{\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$$

$$= \{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 (a\vec{v}_1 + b\vec{v}_2) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$$

$$= \{(\alpha_1 + \alpha_3 a)\vec{v}_1 + (\alpha_2 + \alpha_3 b)\vec{v}_2 \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$$

$$= \{ c\vec{v}_1 + d\vec{v}_2 \mid c, d \in \mathbb{R} \}$$

$$= \text{span}\{\vec{v}_1, \vec{v}_2\}.$$

Q.E.D.

Theorem: Let V be an n-dimensional vector space, and let S be a set ~~with~~ in V with n vectors. Then S is a basis for V iff

S spans V

or S is linearly independent.