Math 22A
Kouba
Properties of Dot Product, Norm

THEOREM 3.2.1 *If $\mathbf{v}$ is a vector in R^n, and if k is any scalar, then:*

(a) $\|\mathbf{v}\| \geq 0$

(b) $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$

(c) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

THEOREM 3.2.2 *If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n, and if k is a scalar, then:*

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **[Symmetry property]**

(b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**

(c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ **[Homogeneity property]**

(d) $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ **[Positivity property]**

THEOREM 3.2.3 *If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n, and if k is a scalar, then:*

(a) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(c) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(d) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

(e) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

THEOREM 3.2.4 Cauchy–Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\| \tag{22}$$

or in terms of components

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \leq (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}(v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} \tag{23}$$

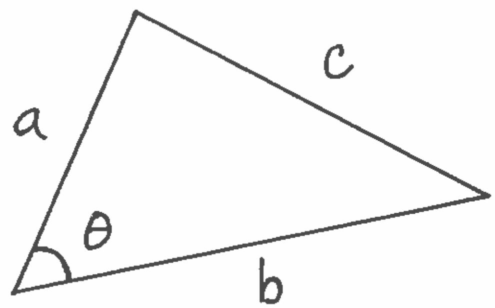
THEOREM 3.2.5 *If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n, then:*

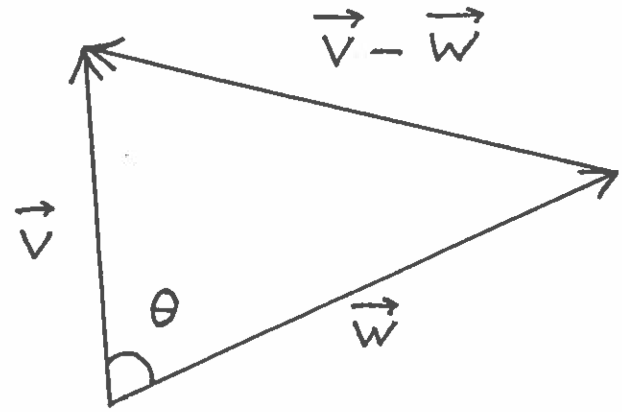
(a) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ **[Triangle inequality for vectors]**

(b) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ **[Triangle inequality for distances]**

Math 22A
Kouba
An Alternate Formula for Dot Product

<u>RECALL:</u> (Law of Cosines) For any triangle $c^2 = a^2 + b^2 - 2ab\cos\theta$.





<u>RECALL:</u> If $\vec{v}$ is a vector in R^n, then $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

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Let $\vec{v}$ and $\vec{w}$ be two vectors in R^n, and consider the triangle formed by the three vectors $\vec{v}$, $\vec{w}$, and $\vec{v} - \vec{w}$. Let θ be the angle between the vectors $\vec{v}$ and $\vec{w}$. By the Law of Cosines we get

$$\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta \quad \longrightarrow$$

$$(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta \quad \longrightarrow$$

$$\vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{w} - \vec{w} \cdot \vec{v} + \vec{w} \cdot \vec{w} = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta \quad \longrightarrow$$

$$\|\vec{v}\|^2 - \vec{v} \cdot \vec{w} - \vec{v} \cdot \vec{w} + \|\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta \quad \longrightarrow$$

$$-2(\vec{v} \cdot \vec{w}) = -2\|\vec{v}\|\|\vec{w}\|\cos\theta \quad \longrightarrow$$

$$\boxed{\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta}$$

or

$$\boxed{\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}}$$