Math 22A
Kouba
Gram-Schmidt Process– Creating an Orthogonal Basis for a Vector Space

DEFINITION: The vector space R^n together with the "dot product" operation is called a **real inner product space**.

DEFINITION: A set S of two or more vectors in a real inner product space is called **orthogonal** if all pairs of distinct vectors are orthogonal, i.e., if $\vec{v} \cdot \vec{w} = 0$ for all $\vec{v}, \vec{w} \in S$ and $\vec{v} \neq \vec{w}$. If each vector in an orthogonal set S has norm (length) 1, then S is called **orthonormal** .

RECALL: If $\vec{v}$ is a nonzero vector in a vector space, then

I.) $\vec{u} = \dfrac{1}{\|\vec{v}\|}\vec{v}$ is a unit vector (length 1) in the same direction as $\vec{v}$.

II.) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

THEOREM: If $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

PROOF: Let (#) $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 + \cdots + k_n\vec{v}_n = \vec{0} \longrightarrow$

(Do $\vec{v}_1$ dot product to both sides of the equation (#) and use the orthogonality of vectors.)

$$\vec{v}_1 \cdot \{k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 + \cdots + k_n\vec{v}_n\} = \vec{v}_1 \cdot \vec{0} \longrightarrow$$

$$k_1(\vec{v}_1 \cdot \vec{v}_1) + k_2(\vec{v}_1 \cdot \vec{v}_2) + k_3(\vec{v}_1 \cdot \vec{v}_3) + \cdots + k_n(\vec{v}_1 \cdot \vec{v}_n) = 0 \longrightarrow$$

$$k_1\|\vec{v}_1\|^2 + k_2(0) + k_3(0) + \cdots + k_n(0) = 0 \longrightarrow$$

$$k_1\|\vec{v}_1\|^2 = 0 \longrightarrow k_1 = 0\ .$$

(Now do $\vec{v}_2$ dot product to both sides of the equation (#) and use the orthogonality of vectors.)

$$\vec{v}_2 \cdot \{k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 + \cdots + k_n\vec{v}_n\} = \vec{v}_2 \cdot \vec{0} \longrightarrow$$

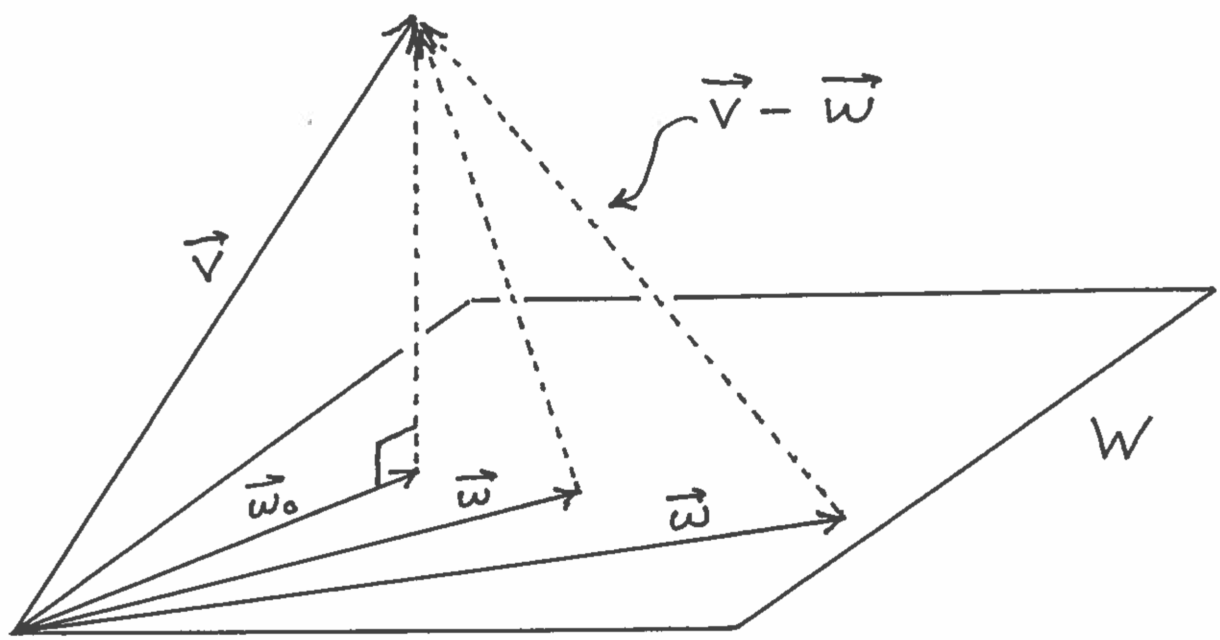
$$k_1(\vec{v}_2 \cdot \vec{v}_1) + k_2(\vec{v}_2 \cdot \vec{v}_2) + k_3(\vec{v}_2 \cdot \vec{v}_3) + \cdots + k_n(\vec{v}_2 \cdot \vec{v}_n) = 0 \longrightarrow$$

$$k_1(0) + k_2\|\vec{v}_2\|^2 + k_3(0) + \cdots + k_n(0) = 0 \longrightarrow$$

$$k_2 \left\| \vec{v}_2 \right\|^2 = 0 \qquad \longrightarrow \qquad k_2 = 0 \ .$$

Continuing in this manner, we can conclude that $k_1 = 0, k_2 = 0, k_3 = 0, \cdots, k_n = 0$. Thus, set S is linearly independent. QED

PROJECTION THEOREM: Consider a subspace W in a vector space V, and let $\vec{v} \in V$. Now consider all vectors of the form $\vec{v} - \vec{w}$, where $\vec{w} \in W$ (See diagram.).



There exists a unique vector $\vec{w}_0 \in W$ so that

$$\left\| \vec{v} - \vec{w}_0 \right\| = \min_{\vec{w} \in W} \left\| \vec{v} - \vec{w} \right\|$$

and $\vec{v} - \vec{w}_0 \in W^{\perp}$.

Summary: For each $\vec{v} \in V$ there is a unique $\vec{w}_0 \in W$, with $\vec{v} - \vec{w}_0 \in W^{\perp}$, so that

$$\vec{v} = (\vec{v} - \vec{w}_0) + \vec{w}_0$$

NOTE: In the following theorems the notation " $\langle \vec{u}, \vec{v} \rangle$ " means the same as " $\vec{u} \cdot \vec{v}$ " (ordinary dot product).

THEOREM 6.3.2

(*a*) *If* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is an orthogonal basis for an inner product space* V, *and if* $\mathbf{u}$ *is any vector in* V, *then*

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n \tag{3}$$

(*b*) *If* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is an orthonormal basis for an inner product space* V, *and if* $\mathbf{u}$ *is any vector in* V, *then*

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n \tag{4}$$

THEOREM 6.3.4 *Let* W *be a finite-dimensional subspace of an inner product space* V.

(*a*) *If* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *is an orthogonal basis for* W, *and* $\mathbf{u}$ *is any vector in* V, *then*

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r \tag{12}$$

(*b*) *If* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *is an orthonormal basis for* W, *and* $\mathbf{u}$ *is any vector in* V, *then*

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r \tag{13}$$

THEOREM 6.3.5 *Every nonzero finite-dimensional inner product space has an orthonormal basis.*

The Gram–Schmidt Process

To convert a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following computations:

Step 1. $\mathbf{v}_1 = \mathbf{u}_1$

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$

Step 4. $\mathbf{v}_4 = \mathbf{u}_4 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$

$\vdots$

(continue for r steps)

Optional Step. To convert the orthogonal basis into an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_r\}$, normalize the orthogonal basis vectors.