Section 2.2

2.) $A=\begin{bmatrix}-6 & 1\\ 2 & -2\end{bmatrix}$, $A^T=\begin{bmatrix}-6 & 2\\ 1 & -2\end{bmatrix}$ then

$\det A = 12-2=10$; $\det A^T = 12-2=10$

3.) $A=\begin{bmatrix}2 & -1 & 3\\ 1 & 2 & 4\\ 5 & -3 & 6\end{bmatrix}$, $A^T=\begin{bmatrix}2 & 1 & 5\\ -1 & 2 & -3\\ 3 & 4 & 6\end{bmatrix}$ then

$\det A = 2(12+12)-1(-6+9)+5(-4-6)$
$= 48-3-50=-5$;

$\det A^T = 3(-3-10)-4(-6+5)+6(4+1)$
$= -39+4+30 = -5$

6.) $\det A = 1$

7.) $\det A = -1$

9.) $\begin{vmatrix}3 & -6 & 9\\ -2 & 7 & -2\\ 0 & 1 & 5\end{vmatrix} = \begin{vmatrix}3 & -6 & 9\\ 1 & 1 & 7\\ 0 & 1 & 5\end{vmatrix} = \begin{vmatrix}0 & -9 & -12\\ 1 & 1 & 7\\ 0 & 1 & 5\end{vmatrix} = (-3)\begin{vmatrix}0 & 3 & 4\\ 1 & 1 & 7\\ 0 & 1 & 5\end{vmatrix}$

$= (-3)\begin{vmatrix}0 & 0 & -11\\ 1 & 1 & 7\\ 0 & 1 & 5\end{vmatrix} = (-3)(-1)\begin{vmatrix}1 & 1 & 7\\ 0 & 0 & -11\\ 0 & 1 & 5\end{vmatrix} = (3)(-1)\begin{vmatrix}1 & 1 & 7\\ 0 & 1 & 5\\ 0 & 0 & -11\end{vmatrix}$

$= (-3)(-11) = 33$

14.) $\begin{vmatrix}1 & -2 & 3 & 1\\ 5 & -9 & 6 & 3\\ -1 & 2 & -6 & -2\\ 2 & 8 & 6 & 1\end{vmatrix} = \begin{vmatrix}1 & -2 & 3 & 1\\ 0 & 1 & -9 & -2\\ 0 & 0 & -3 & -1\\ 0 & 12 & 0 & -1\end{vmatrix} = \begin{vmatrix}1 & -2 & 3 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & -3 & -1\\ 0 & 12 & 0 & -1\end{vmatrix}$

$$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -13 \end{vmatrix} = (1)(1)(-3)(-13) = 39$$

15.) $\det A = -6$ (since 2 row exchanges)

16.) $\det A = 6$ (since 1 row exchange)

17) $\det A = 72$ (factor out $3, -1, 4$)

18.) $\det A = 6$ (factor out -1)

21.) $\det A = 18$ (factor out -3)

22.) $\det A = 0$ (since 2 rows are multiples)

24.) a.) $\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1) \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{13} \end{vmatrix}$

$= -a_{13} a_{22} a_{31}$

29.) Columns 2 and 4 are multiples

30.) add rows 2, 3, 4, 5 to row 1 getting a row of zeroes

35.)

$$\begin{array}{r} (x)(-1)(+) \\ (x)(-1)(+) \\ (x)(-1)(+) \\ \end{array} \begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} = \begin{array}{c} \begin{array}{ccc} (+) & (+) & (+) \end{array} \\ \begin{vmatrix} a & b & b & b \\ b-a & a-b & 0 & 0 \\ 0 & b-a & a-b & 0 \\ 0 & 0 & b-a & a-b \end{vmatrix} \end{array}$$

$$= \begin{vmatrix} a+3b & 3b & 2b & b \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{vmatrix} = (a+3b)(a-b)^3$$

TRUE/FALSE

(a) T (b) T (c) F (d) F

(e) T (f) T

Section 2.3

2.) $A = \begin{bmatrix} 2 & 2 \\ 5 & -2 \end{bmatrix} \Rightarrow \det A = (2)(-2) - (2)(5)$

$= -14$;

$\det(kA) = \det\left(-4 \begin{bmatrix} 2 & 2 \\ 5 & -2 \end{bmatrix}\right)$

$= \det \begin{bmatrix} -8 & -8 \\ -20 & 8 \end{bmatrix} = (-8)(8) - (-8)(-20)$

$= -64 - 160 = -224 = 16(-14)$

$= (-4)^2(-14) = k^2 \det A$

4.) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \det A = (1)\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}$

$- (0)\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} + (0)\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$

$= -4 - 3 = -7$;

$\det(kA) = \det\left(3 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}\right)$

$= \det \begin{bmatrix} 3 & 3 & 3 \\ 0 & 6 & 9 \\ 0 & 3 & -6 \end{bmatrix} = (3)\begin{vmatrix} 6 & 9 \\ 3 & -6 \end{vmatrix} - (0)\begin{vmatrix} 3 & 3 \\ 3 & -6 \end{vmatrix}$

$$+(0)\begin{vmatrix}3 & 3\\6 & 9\end{vmatrix} = 3(-36-27) = -189$$

$$= 27(-7) = 3^3(-7) = k^3 \det A$$

5.) $A = \begin{bmatrix}2 & 1 & 0\\3 & 4 & 0\\0 & 0 & 2\end{bmatrix}$, $B = \begin{bmatrix}1 & -1 & 3\\7 & 1 & 2\\5 & 0 & 1\end{bmatrix} \Rightarrow$

$$A+B = \begin{bmatrix}3 & 0 & 3\\10 & 5 & 2\\5 & 0 & 3\end{bmatrix};$$

$$AB = \begin{bmatrix}2 & 1 & 0\\3 & 4 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & -1 & 3\\7 & 1 & 2\\5 & 0 & 1\end{bmatrix} = \begin{bmatrix}9 & -1 & 8\\31 & 1 & 17\\10 & 0 & 2\end{bmatrix} \Rightarrow$$

$$\begin{array}{ccccc} + & + & + & - & - & - \end{array}$$

$$\det(AB) = \left|\begin{matrix}9 & -1 & 8\\31 & 1 & 17\\10 & 0 & 2\end{matrix}\right|\begin{matrix}9 & -1\\31 & 1\\10 & 0\end{matrix}$$

$$= +(18) + (-170) + (0) - (80) - (0) - (-62)$$

$$= -170;$$

$$\det A = (2)\begin{vmatrix}2 & 1\\3 & 4\end{vmatrix} = 2(8-3) = 10;$$

$$\begin{array}{ccccc} + & + & + & - \; - & - \end{array}$$

$$\det B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix} \begin{matrix} 1 & -1 \\ 7 & 1 \\ 5 & 0 \end{matrix}$$

$$= +(1) + (-10) + (0) - (15) - (0) - (-7)$$

$$= -17 \; ;$$

$$BA = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{bmatrix} \Rightarrow$$

$$\begin{array}{ccccc} + & + & + & - \; - & - \end{array}$$

$$\det(BA) = \begin{bmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{bmatrix} \begin{matrix} -1 & -3 \\ 17 & 11 \\ 10 & 5 \end{matrix}$$

$$= +(-22) + (-120) + (510) - (660) - (-20) - (-102)$$

$$= -170 \; ; \quad \text{then} \quad \det(AB) = \det(BA) \; ;$$

$$\det(A+B) = \begin{vmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{vmatrix} = (5) \begin{vmatrix} 3 & 3 \\ 5 & 3 \end{vmatrix}$$

$$= 5(9-15) = -30 \; ; \quad \text{so}$$

$$\det(A+B) \neq \det A + \det B$$

7.) $\begin{vmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{vmatrix} = -(-1)\begin{vmatrix} 5 & 5 \\ 4 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix}$

$- (0)\begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix}$

$= (15-20) - (6-10) = -5+4 = -1 \neq 0,$

so A is invertible

8.) $\begin{vmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{vmatrix} = -(0)\begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} + (3)\begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix}$

$-(0)\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix}$

$= 3(-8+6) = -6 \neq 0$, so A is invertible

11.) $\begin{vmatrix} 4 & 2 & 8 \\ -2 & 1 & -4 \\ 3 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 2(4) \\ -2 & 1 & 2(-2) \\ 3 & 1 & 2(3) \end{vmatrix} = 0,$

so A is not invertible

15) $\begin{vmatrix} k-3 & -2 \\ -2 & k-2 \end{vmatrix} = (k-3)(k-2) - (-2)(-2)$

$= k^2 - 3k - 2k + 6 - 4$

$= k^2 - 5k + 2 \neq 0 \Rightarrow$

$k \neq \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(2)}}{2(1)} = \dfrac{5 \pm \sqrt{17}}{2}$

$\Rightarrow$ A is invertible

16.) $\begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix} = k^2 - 4 = (k-2)(k+2) \neq 0$

$\Rightarrow k \neq 2,\ k \neq -2 \Rightarrow$ A is invertible

17.) $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{vmatrix} = +(1)\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} - (3)\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} + k\begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix}$

$= (2-18) - 3(4-12) + k(12-4)$

$= -16 + 24 + 8k = 8k + 8 \neq 0 \Rightarrow$

$k \neq -1 \Rightarrow$ A is invertible

18.) $\begin{vmatrix} 1 & 2 & 0 \\ k & 1 & k \\ 0 & 2 & 1 \end{vmatrix} = +(1)\begin{vmatrix} 1 & k \\ 2 & 1 \end{vmatrix} - (2)\begin{vmatrix} k & k \\ 0 & 1 \end{vmatrix}$

$$+(0)\begin{vmatrix} k & 1 \\ 0 & 2 \end{vmatrix} = (1-2k) - 2(k-0)$$

$$= 1-4k \neq 0 \Rightarrow k \neq \tfrac{1}{4} \Rightarrow$$

A is invertible

30.) $\begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$$= +(0)\begin{vmatrix} \sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} - (0)\begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & 0 \end{vmatrix}$$

$$+(1)\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$= \cos^2\theta - (-\sin^2\theta)$$

$$= \cos^2\theta + \sin^2\theta = 1 \neq 0 \Rightarrow$$

A is invertible

34.) A is 4×4 and $\det A = -2$:

a.) $\det(-A) = \det((-1)A)$

$$= (-1)^4 \cdot \det A = (1)(-2) = -2$$

b.) $\det(A^{-1}) = \dfrac{1}{\det A} = \dfrac{1}{-2} = \dfrac{-1}{2}$

c.) $\det(2A^T) = 2^4 \det(A^T)$

$= 16 \det A = 16(-2) = -32$

d.) $\det(A^3) = (\det A)(\det A)(\det A)$

$= (\det A)^3 = (-2)^3 = -8$

35.) A is 3×3 and $\det A = 7$:

a.) $\det(3A) = 3^3 \det A = 27(7) = 189$

b.) $\det(A^{-1}) = \dfrac{1}{\det A} = 1/7$

c.) $\det(2A^{-1}) = 2^3 \cdot \det(A^{-1})$

$= 8 \cdot 1/7 = 8/7$

d.) $\det((2A)^{-1}) = \det(\frac{1}{2}A^{-1})$

$= \left(\frac{1}{2}\right)^3 \det A^{-1}$

$= \dfrac{1}{8} \cdot \dfrac{1}{7} = \dfrac{1}{56}$

36.) Square matrix A is invertible iff A^TA is invertible:

<u>proof</u>: ($\Rightarrow$) assume A is invertible. Show A^TA is invertible. But A invertible $\Rightarrow$ $\det A \neq 0$. Then

$$\det(A^TA) = \det(A^T)\cdot \det A$$
$$= (\det A)(\det A)$$
$$= (\det A)^2 \neq 0 \Rightarrow$$

A^TA is invertible.

($\Leftarrow$) assume A^TA is invertible. Show A is invertible. But A^TA invertible $\Rightarrow$

$$\det(A^TA) \neq 0 \Rightarrow$$
$$(\det(A^T))(\det A) = (\det A)(\det A)$$
$$= (\det A)^2 \neq 0 \Rightarrow$$

$\det A \neq 0 \Rightarrow A$ is invertible.

Q.E.D.

37.) If A is square, then $\det(A^T A) = \det(AA^T)$:

proof: $\det(A^T A) = (\det(A^T))(\det A)$

$= (\det A)(\det A)$

$= (\det A)(\det(A^T))$

$= \det(AA^T)$. Q.E.D.

TRUE/FALSE

(a) F (b) F (c) T

(d) F (i) T (j) T