Section 4.2

1.) a.) $W = \{(a,0,0) \mid (a,0,0) \in \mathbb{R}^3\}$:

$W \neq \emptyset$ since $(1,0,0) \in W$

i.) If $\vec{u} = (a,0,0), \vec{v} = (b,0,0) \in W$, then

$\vec{u} + \vec{v} = (a,0,0) + (b,0,0) = (a+b,0,0) \in \mathbb{R}^3$

ii.) If $\vec{u} = (a,0,0)$ and $k \in \mathbb{R}$, then

$k\vec{u} = k(a,0,0) = (ka,0,0) \in \mathbb{R}^3$,

so W is a subspace of $\mathbb{R}^3$

b.) $W = \{(a,1,1) \mid (a,1,1) \in \mathbb{R}^3\}$:

$W \neq \emptyset$ since $(0,1,1) \in W$

i.) If $\vec{u} = (a,1,1), \vec{v} = (b,1,1) \in W$, then

$\vec{u} + \vec{v} = (a,1,1) + (b,1,1) = (a+b,2,2) \notin W$,

so W is NOT a subspace of $\mathbb{R}^3$

c.) $W = \{(a,b,c) \mid (a,b,c) \in \mathbb{R}^3 \text{ and } b = a+c\}$:

$W \neq \emptyset$ since $(2,5,3) \in W$ because

$2+3=5$

i.) If $\vec{u} = (a_1,b_1,c_1), \vec{v} = (a_2,b_2,c_2) \in W$, then

$b_1 = a_1 + c_1$ and $b_2 = a_2 + c_2$ and

$\vec{u} + \vec{v} = (a_1,b_1,c_1) + (a_2,b_2,c_2)$

$= (a_1+a_2, b_1+b_2, c_1+c_2) \in W$

since $b_1+b_2 = (a_1+c_1)+(a_2+c_2)$
$= (a_1+a_2)+(c_1+c_2)$

ii.) If $\vec{u}=(a,b,c)\in W$ and $k\in\mathbb{R}$ then $b=a+c$,
and $k\vec{u}=k(a,b,c)=(ka,kb,kc)\in W$
since $kb=k(a+c)=ka+kc$,
so W is a subspace of $\mathbb{R}^3$

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d.) $W=\{(a,b,c) \mid (a,b,c)\in\mathbb{R}^3 \text{ and } b=a+c+1\}$:
$W\neq\phi$ since $(2,6,3)\in W$ because
$6=2+3+1$

i.) If $\vec{u}=(a_1,b_1,c_1), \vec{v}=(a_2,b_2,c_2)\in W$, then
$b_1=a_1+c_1+1$ and $b_2=a_2+c_2+1$, but
$\vec{u}+\vec{v}=(a_1,b_1,c_1)+(a_2,b_2,c_2)$
$=(a_1+a_2,\ b_1+b_2,\ c_1+c_2)\notin W$ since
$b_1+b_2=(a_1+c_1+1)+(a_2+c_2+1)$
$=(a_1+a_2)+(c_1+c_2)+2$
$\neq(a_1+a_2)+(c_1+c_2)+1$,
so W is NOT a subspace of $\mathbb{R}^3$

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e.) $W=\{(a,b,0) \mid (a,b,0)\in\mathbb{R}^3\}$:
$W\neq\phi$ since $(1,2,0)\in W$

i.) If $\vec{u} = (a,b,0)$, $\vec{v} = (c,d,0) \in W$, then
$\vec{u}+\vec{v} = (a,b,0)+(c,d,0) = (a+c, b+d, 0) \in W$

ii.) If $\vec{u} = (a,b,0) \in W$ and $k \in \mathbb{R}$, then
$k\vec{u} = k(a,b,0) = (ka, kb, 0) \in W$,
so W is a subspace of $\mathbb{R}^3$

2.) a.) $W = \left\{ \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots & a_n \end{bmatrix} \middle| \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots & a_n \end{bmatrix} \in M_{nn} \right\}$:

$W \neq \emptyset$ since $\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{bmatrix} \in W$

i.) If $\vec{u} = \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots & a_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} b_1 & & 0 \\ & b_2 & \\ 0 & & \ddots & b_n \end{bmatrix} \in W$, then

$$\vec{u}+\vec{v} = \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots & a_n \end{bmatrix} + \begin{bmatrix} b_1 & & 0 \\ & b_2 & \\ 0 & & \ddots & b_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1+b_1 & & 0 \\ & a_2+b_2 & \\ & & \ddots \\ 0 & & & a_n+b_n \end{bmatrix} \in W$$

ii.) If $\vec{u} = \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots & a_n \end{bmatrix} \in W$ and $k \in \mathbb{R}$, then

$$k\vec{u} = k\begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix} = \begin{bmatrix} ka_1 & & & 0 \\ & ka_2 & & \\ & & \ddots & \\ 0 & & & ka_n \end{bmatrix} \in W,$$

so W is a subspace of M_{nn}

b.) $W = \{A \in M_{nn} \mid \det A = 0\}$:

$W \neq \emptyset$ since $\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 0 \end{bmatrix} \in W$

because $\det \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 0 \end{bmatrix} = (1)(1)\cdots(1)(0) = 0$

i.) $\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & & & 0 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 0 & & & & 1 \end{bmatrix} \in W$

since $\det \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 0 \end{bmatrix} = 0$ and

$\det \begin{bmatrix} 0 & & & & 0 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 0 & & & & 1 \end{bmatrix} = 0$, but

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & 0 \\ & 0 & & \\ & & \ddots & \\ & & 0 & \\ 0 & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \notin W$$

since $\det \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = (1)(1)\cdots(1) = 1 \neq 0,$

so W is NOT a subspace of M_{nn}

e.) $W = \{A \in M_{nn} \mid A^T = -A\}$:

$W \neq \phi$ since $\vec{0} = \begin{bmatrix} 0 & & 0 \\ & 0 & \\ & \ddots & \\ 0 & & 0 \end{bmatrix} \in W$

because $\vec{0}^T = -\vec{0}$

i.) If $\vec{u} = A, \vec{v} = B \in W$, then

$A^T = -A$, $B^T = -B$, and

$\vec{u} + \vec{v} = A + B \in W$ since

$(A+B)^T = A^T + B^T = -A + -B = -(A+B)$

ii) If $\vec{u} = A \in W$ and $k \in \mathbb{R}$, ~~then~~

$k\vec{u} = kA \in W$ since

$(kA)^T = kA^T = k(-A) = -kA$,

so W is a subspace of M_{nn}

3.) b.) $W = \{ p(x) \in P_3 \mid p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$
and $a_0 + a_1 + a_2 + a_3 = 0 \}$:

$W \neq \phi$ since $1 - x + 2x^2 - 2x^3 \in W$
because $1 + {}^-1 + 2 + {}^-2 = 0$

i.) If $p(x), q(x) \in W$, then
$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and
$q(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ with
$a_0 + a_1 + a_2 + a_3 = 0$ and $b_0 + b_1 + b_2 + b_3 = 0$;
then
$p(x) + q(x) = (a_0 + a_1x + a_2x^2 + a_3x^3) + (b_0 + b_1x + b_2x^2 + b_3x^3)$
$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \in W$

since $(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3)$
$= (a_0 + a_1 + a_2 + a_3) + (b_0 + b_1 + b_2 + b_3) = 0 + 0 = 0$

ii.) If $p(x) \in P_3$ and $k \in \mathbb{R}$, then
$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ with
$a_0 + a_1 + a_2 + a_3 = 0$ and
$kp(x) = k(a_0 + a_1x + a_2x^2 + a_3x^3)$
$= (ka_0 + ka_1 + ka_2 + ka_3) \in W$ since

$ka_0+ka_1+ka_2+ka_3 = k(a_0+a_1+a_2+a_3)$
$= k(0) = 0$, so W is a subspace of P_3

4.) $F(-\infty,\infty) = \{\text{functions } f \mid f' \text{ is continuous on the interval } -\infty < x < \infty\}$;

b) $W = \{f \in F(-\infty,\infty) \mid f(0) = 1\}$:
$W \neq \phi$ since $f(x) = x^2+1 \rightarrow f'(x) = 2x$, which is continuous for $-\infty < x < \infty$, and $f(0) = 1$, so $f \in W$

i.) If $f, g \in W$, then f' and g' are continuous for $-\infty < x < \infty$, so $f'+g' = (f+g)'$ is continuous for $-\infty < x < \infty$; and $f(0)=1$, $g(0)=1$, but $(f+g)(0) = f(0)+g(0) = 1+1 = 2 \neq 1$, so $f+g \notin W$, so W is NOT a subspace of $F(-\infty,\infty)$

5.) $\mathbb{R}^\infty = \{(x_1,x_2,x_3,\ldots) \mid x_i \in \mathbb{R} \text{ for } i=1,2,3,\ldots\}$;

b.) $W = \{(v,1,v,1,v,1,\ldots) \mid v \in \mathbb{R}\}$:
$W \neq \phi$ since $\vec{u} = (0,1,0,1,0,1,\ldots) \in W$

i.) Note that $\vec{u} = (0,1,0,1,0,1,\ldots) \in W$

and $\vec{v} = (2,1,2,1,2,1,\ldots) \in W$, but

$\vec{u}+\vec{v} = (0+2, 1+1, 0+2, 1+1, 0+2, 1+1, \ldots)$

$= (2,2,2,2,2,2,\ldots) \notin W$,

so W is NOT a subspace of $\mathbb{R}^\infty$

c.) $W = \{(v, 2v, 4v, 8v, 16v, \ldots) \mid v \in \mathbb{R}\}$:

$W \neq \phi$ since $\vec{v} = (1,2,4,8,16,\ldots) \in W$

i.) If $\vec{u} = (u, 2u, 4u, 8u, \ldots)$, $\vec{v} = (v, 2v, 4v, 8v, \ldots) \in W$, then

$\vec{u}+\vec{v} = (u+v, 2u+2v, 4u+4v, 8u+8v, \ldots)$

$= ((u+v), 2(u+v), 4(u+v), 8(u+v), \ldots) \in W$

since $u+v \in \mathbb{R}$

ii.) If $\vec{u} = (u, 2u, 4u, 8u, \ldots) \in W$ and $k \in \mathbb{R}$, then $k\vec{u} = (ku, k2u, k4u, k8u, \ldots)$

$= ((ku), 2(ku), 4(ku), 8(ku), \ldots) \in W$

since $ku \in \mathbb{R}$, so W is a subspace of $\mathbb{R}^\infty$

7.) $\vec{u} = \overrightarrow{(0,-2,2)}$, $\vec{v} = (1, 3, -1)$

a.) $(2,2,2) = k_1 \overrightarrow{(0,-2,2)} + k_2 \overrightarrow{(1,3,-1)}$

$= \overrightarrow{(k_2, -2k_1+3k_2, 2k_1-k_2)} \Rightarrow$

$\begin{cases} \boxed{k_2 = 2} \\ \left.\begin{array}{l} -2k_1+3k_2 = 2 \\ 2k_1 - k_2 = 2 \end{array}\right\} \Rightarrow \begin{cases} -2k_1+6 = 2 \\ 2k_1 - 2 = 4 \end{cases} \Rightarrow \end{cases}$

$\begin{cases} k_1 = 2 \\ \boxed{k_1 = 2} \end{cases}$, so YES

b.) $(0,4,5) = k_1 \overrightarrow{(0,-2,2)} + k_2 \overrightarrow{(1,3,-1)}$

$= \overrightarrow{(k_2, -2k_1+3k_2, 2k_1-k_2)} \Rightarrow$

$\begin{cases} k_2 = 0 \\ \left.\begin{array}{l} -2k_1+3k_2 = 4 \\ 2k_1 - k_2 = 5 \end{array}\right\} \Rightarrow \begin{cases} -2k_1 = 4 \\ 2k_1 = 5 \end{cases} \Rightarrow \end{cases}$

$\begin{cases} k_1 = -2 \\ k_1 = 5/2 \end{cases}$, so no solution, so NO

8.) $\vec{u} = \overrightarrow{(2,1,4)}$, $\vec{v} = \overrightarrow{(1,-1,3)}$, $\vec{w} = \overrightarrow{(3,2,5)}$

b.) $\overrightarrow{(6,11,6)} = k_1\vec{u} + k_2\vec{v} + k_3\vec{w} \Rightarrow$

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 6 \\ 1 & -1 & 3 & 11 \\ 3 & 2 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 3 & -2 & -16 \\ 1 & -1 & 3 & 11 \\ 0 & 5 & -4 & -27 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 3 & -2 & -16 \\ 1 & -1 & 3 & 11 \\ 0 & -1 & 0 & 5 \end{array}\right]$$

$$\sim \begin{bmatrix} 0 & 0 & -2 & -1 \\ 1 & 0 & 3 & 6 \\ 0 & -1 & 0 & 5 \end{bmatrix} \sim \begin{array}{cccc} k_1 & k_2 & k_3 & \\ \end{array} \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1 & 0 & 0 & 9/2 \\ 0 & 1 & 0 & -5 \end{bmatrix} \Rightarrow$$

$k_1 = 9/2,\ k_2 = -5,\ k_3 = 1/2$

9.) $A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix},\ B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix},\ C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$

a.) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = k_1 A + k_2 B + k_3 C \Rightarrow$

$$\begin{bmatrix} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{bmatrix} \sim \begin{bmatrix} 0 & 5 & 2 & 4 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ 0 & 1 & 3 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & -13 & 39 \\ 0 & 0 & 5 & -15 \\ -2 & 0 & -5 & 13 \\ 0 & 1 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -3 \\ -2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\begin{array}{ccc} k_1 & k_2 & k_3 \end{array}$$
$$\sim \begin{bmatrix} 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \Rightarrow k_1 = 1,\ k_2 = 2,\ k_3 = -3,$$

~~so~~ YES

c.) $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix} = k_1 A + k_2 B + k_3 C \Rightarrow$

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & -1 \\ 0 & -1 & 2 & 5 \\ -2 & 2 & 1 & 7 \\ -2 & 3 & 4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 5 & 2 & 13 \\ 0 & -1 & 2 & 5 \\ -2 & 2 & 1 & 7 \\ 0 & 1 & 3 & -6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & -13 & 43 \\ 0 & 0 & 5 & -1 \\ -2 & 0 & -5 & 19 \\ 0 & 1 & 3 & 6 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & 1 & -43/13 \\ 0 & 0 & 1 & -1/5 \\ -2 & 0 & -5 & 19 \\ 0 & 1 & 3 & 6 \end{array}\right] \end{array} \Rightarrow$$

$k_3 = {}^{-43}/_{15}$ and $k_3 = {}^{-1}/_{5}$ so no solution, so NO

10.) $\vec{p_1} = 2 + x + 4x^2$, $\vec{p_2} = 1 - x + 3x^2$,
$\vec{p_3} = 3 + 2x + 5x^2$

a.) $-9 - 7x - 15x^2 = k_1\vec{p_1} + k_2\vec{p_2} + k_3\vec{p_3} \Rightarrow$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 3 & -1 & 5 \\ 1 & -1 & 2 & -7 \\ 0 & 7 & -3 & 13 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 3 & -1 & 5 \\ 1 & -1 & 2 & -7 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 2 & -4 \\ 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right] \Rightarrow$$

$k_1 = -2$, $k_2 = 1$, $k_3 = {}^{-}2$

11.) a.)

$$\det\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} = 2(0-3)-2(0-0)+2(0-0)$$
$$= -6 \neq 0,$$

so YES spans $\mathbb{R}^3$

b.)
$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & 2 \\ 8 & -1 & 8 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -4 \\ 0 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix},$$

so span of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is equal to span of $\overrightarrow{(2,-1,3)}, \overrightarrow{(0,3,-4)}$, so NO does not span $\mathbb{R}^3$

12.) a.)
$$\left[\begin{array}{cccc} -1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 3 \\ 3 & -1 & 5 & 2 \\ \hline 2 & 3 & -7 & 3 \end{array}\right] \sim \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & 1 & 4 & 5 \\ 0 & -1 & 11 & 5 \\ 0 & 3 & -3 & 5 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 15 & 10 \\ 0 & 0 & -15 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 15 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $\vec{v} = \overrightarrow{(2,3,-7,3)}$ is in the span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ (YES)

c.)
$$\left[\begin{array}{cccc} -1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 3 \\ 3 & -1 & 5 & 2 \\ \hline 1 & 1 & 1 & 1 \end{array}\right] \sim \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & 1 & 4 & 5 \\ 0 & -1 & 11 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 15 & 10 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & -35 \\ 0 & 0 & 1 & 3 \end{bmatrix},$$ so $\vec{v} = \overrightarrow{(1,1,1,1)}$ and $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent $\Rightarrow$ $\vec{v}$ is NOT in span$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ (NO)

13.) $P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 5 & -1 & 4 \\ -2 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -6 \\ 0 & 4 & -6 \\ 0 & -4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$ so

$$\text{span}\{p_1, p_2, p_3, p_4\} = \text{span}\{1 - x + 2x^2, 4x - 6x^2\} \neq P_2 \quad (\text{NO})$$

14.) a.) $\cos 2x = \cos^2 x - \sin^2 x$
$= (1)\cos^2 x + (-1)\sin^2 x$ (YES)

b.) $3 + x^2 \neq (a)\cos^2 x + (b)\sin^2 x$ (NO)

c.) $1 = \cos^2 x + \sin^2 x = (1)\cos^2 x + (1)\sin^2 x$ (YES)

d.) $\sin x \neq (a)\cos^2 x + (b)\sin^2 x$ (NO) e.) $0 = (0)\cos^2 x + (0)\sin^2 x$ (YES)

15.) a.) $A\vec{x} = \vec{0} \Rightarrow$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & -4 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & -2 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{cases} x_1 + \frac{1}{2}x_3 = 0 \\ x_2 + \frac{3}{2}x_3 = 0 \end{cases}, \text{ so let}$$

$x_3 = t$ any # $\Rightarrow x_1 = -\frac{1}{2}t$, $x_2 = -\frac{3}{2}t$, then

solution is a <u>line</u>

$$L: \begin{cases} x_1 = -\frac{1}{2}t \\ x_2 = -\frac{3}{2}t \\ x_3 = t \end{cases}$$

b.) $A\vec{x} = \vec{0} \Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ 1 & 0 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -2 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 9 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array} \Rightarrow x_1 = 0,\ x_2 = 0,\ x_3 = 0 \text{ so}$$

<u>origin</u> <u>only</u> solution

c.) $A\vec{x} = \vec{0} \Longrightarrow$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Longrightarrow$$

$x_1 - 3x_2 + x_3 = 0$, so solution is a <u>plane</u>.

17.) $C[a,b] = \{f \mid f$ is a continuous function on $[a,b]\}$:

$W = \{f \in C[a,b] \mid \int_a^b f(x)\,dx = 0\}$;

$W \neq \phi$ since $f(x) = 0 \in W$ because

$\int_a^b f(x)\,dx = \int_a^b 0\,dx = c\Big|_a^b = c - c = 0$

i.) If $f, g \in W$, then $\int_a^b f(x)\,dx = 0$ and $\int_a^b g(x)\,dx = 0$, so that

$\int_a^b (f+g)(x)\,dx = \int_a^b (f(x) + g(x))\,dx$

$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$= 0 + 0 = 0$; thus $f+g \in W$

ii) If $f \in W$ and $k \in \mathbb{R}$, then $\int_a^b f(x)\,dx = 0$ and $\int_a^b (kf)(x)\,dx$

$= \int_a^b k \cdot f(x)\,dx = k \int_a^b f(x)\,dx$

$= k \cdot (0) = 0$, so that $kf \in W$; thus, W is a subspace of $C[a,b]$

19.) b.) $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$, $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$T\vec{v} = A\vec{v}$; if $\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

then $T\vec{u}_1 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$,

$T\vec{u}_2 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, but

$\begin{bmatrix} -2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, so $\text{span}\{T\vec{u}_1, T\vec{u}_2\}$

$= \text{span}\{\overrightarrow{(-1,2)}\} \neq \mathbb{R}^2$

(NOTE: $\overrightarrow{(a,b)} = \begin{bmatrix} a \\ b \end{bmatrix}$.)

20.) a.) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $T: \mathbb{R}^3 \to \mathbb{R}^2$ by

$T\vec{v} = A\vec{v}$; if $\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

then $T\vec{u}_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$T\vec{u}_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$T\vec{u}_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, ~~then~~

$\text{span}\{T\vec{u}_1, T\vec{u}_2, T\vec{u}_3\}$

$= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$

$= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$

$\left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$

(NOTE: $\overrightarrow{(a,b)} = \begin{bmatrix} a \\ b \end{bmatrix}$.)

TRUE / FALSE

(a) T (b) T (c) F (d) F
(e) F (f) T (g) T (h) F
(i) F (j) T (k) F