Section 4.3

1.) a.) $(5)\overrightarrow{(-1,2,4)} + (1)\overrightarrow{(5,-10,-20)}$

$= \overrightarrow{(-5,10,20)} + \overrightarrow{(-5,-10,-20)}$

$= \overrightarrow{(0,0,0)}$, so lin. dep.

b.) $a\overrightarrow{(3,-1)} + b\overrightarrow{(4,5)} + c\overrightarrow{(-4,7)} = \overrightarrow{(0,0)}$

$\Rightarrow \begin{cases} 3a + 4b - 4c = 0 \\ -a + 5b + 7c = 0 \end{cases}$

$\Rightarrow 19b + 17c = 0$ so let $c = -1 \Rightarrow$

$19b - 17 = 0 \Rightarrow 19b = 17 \Rightarrow b = 17/19$

and $a = 5b + 7c = 5\left(\frac{17}{19}\right) + 7(-1)$

$= \frac{85}{19} - \frac{133}{19} = \frac{-48}{19}$, so

$\left(-\frac{48}{19}\right)\overrightarrow{(3,-1)} + \left(\frac{17}{19}\right)\overrightarrow{(4,5)} + (-1)\overrightarrow{(-4,7)} = \overrightarrow{(0,0)}$,

and lin. dep.

c.) $(-2)(3-2x+x^2) + (1)(6-4x+2x^2)$ -

$= -6 + 4x - 2x^2 + 6 - 4x + 2x^2 = 0$,

so lin. dep.

d.) $(1)\begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix} + (1)\begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

so lin. dep.

2.) a.) $\det \begin{bmatrix} -3 & 0 & 4 \\ 5 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix} = -3(-3-2) - 0(15-2) + 4(5+1)$

$= 15 - 0 + 24 = 39 \neq 0$, so vectors are lin. ind.

b.) $\begin{bmatrix} -2 & 0 & 1 \\ 3 & 2 & 5 \\ 6 & -1 & 1 \\ 7 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 3 & 2 & 5 \\ 6 & -1 & 1 \\ 1 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 0 & -4 & -13 \\ 0 & -5 & -9 \\ 0 & 0 & -9 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so $\overrightarrow{(6,-1,1)}$

is a linear combination of the other 3 vectors, so lin. dep.

3.) a.) $\begin{bmatrix} 1 & 5 & 3 & -1 \\ 3 & 8 & 7 & -3 \\ 2 & -1 & 2 & 6 \\ 4 & 2 & 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 & -1 \\ 0 & -7 & -2 & 0 \\ 0 & -11 & -4 & 8 \\ 0 & 4 & 2 & -8 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 5 & 3 & -1 \\ 0 & -7 & -2 & 0 \\ 0 & 1 & 2 & -16 \\ 0 & 2 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 & -1 \\ 0 & -1 & 1 & -12 \\ 0 & 1 & 2 & -16 \\ 0 & 0 & -3 & 28 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 5 & 3 & -1 \\ 0 & 0 & 3 & -28 \\ 0 & 1 & 2 & -16 \\ 0 & 0 & -3 & 28 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 & -1 \\ 0 & 0 & 3 & -28 \\ 0 & 1 & 2 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ so

$\overrightarrow{(4,2,6,11)}$ is a linear combination of the other 3 vectors, so lin. dep.

4.) a.)

$$\det \begin{bmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{bmatrix} = 2(-24-20)-(-1)(-12-4) + 4(30-12)$$

$= -88-16+72 = -32 \neq 0$, so lin. ind.

5.) a.) $a\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} a+b=0 \\ 2b+c=0 \longrightarrow 2(-c)+c=0 \\ a+2b+2c=0 \\ 2a+b+c=0 \end{cases}$$

$\left.\begin{matrix} a+2b+2c=0 \\ 2a+b+c=0 \end{matrix}\right\} \Rightarrow -3b-3c=0$

$\Rightarrow b=-c$

$-c=0 \Rightarrow \boxed{c=0}$, $\boxed{b=0}$ and

$a+b=0 \Rightarrow a+0=0 \Rightarrow \boxed{a}$, so lin. ind.

b.) $a\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}+b\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}+c\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$

$\Rightarrow a=0,\ b=0,\ c=0,$ ~~so~~
lin. ind.

6.) $a\begin{bmatrix}1&0\\1&k\end{bmatrix}+b\begin{bmatrix}-1&0\\k&1\end{bmatrix}+c\begin{bmatrix}2&0\\1&3\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}\Rightarrow$

$\begin{cases}a-b+2c=0\\a+bk+c=0\\ak+b+3c=0\end{cases}$ and for solution $a=0, b=0, c=0$

we need $\det\begin{bmatrix}1&-1&2\\1&k&1\\k&1&3\end{bmatrix}\neq 0\Rightarrow$

$1(3k-1)-(-1)(3-k)+2(1-k^2)\neq 0\Rightarrow$

$3k-1+3-k+2-2k^2\neq 0\Rightarrow$

$-2k^2+2k+4\neq 0\Rightarrow$

$k^2-k-2\neq 0\Rightarrow$

$(k-2)(k+1)\neq 0\Rightarrow$ vectors are linearly independent if $k\neq 2,\ k\neq -1$.

7.) These vectors lie in the same plane iff they are linearly dependent:

a.)

$$\det\begin{bmatrix} 2 & -2 & 0 \\ 6 & 1 & 4 \\ 2 & 0 & -4 \end{bmatrix} = 2(-4-0)-(-2)(-24-8) + 0(0-2)$$

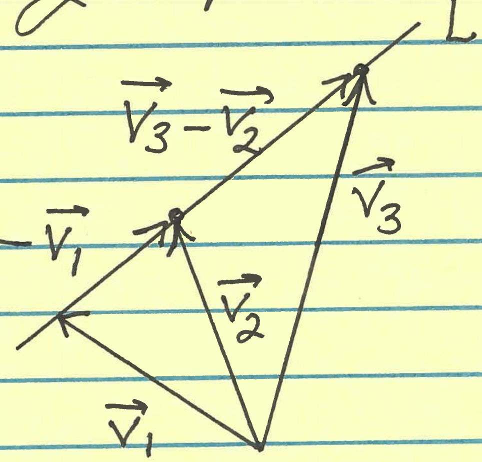
$= -8-64 = -72 \neq 0$, so vectors are lin. ind. $\Rightarrow$ NOT in the same plane

b.)

$$\det\begin{bmatrix} -6 & 7 & 2 \\ 3 & 2 & 4 \\ 4 & -1 & 2 \end{bmatrix} = -6(4+4)-7(6-16) + 2(-3-8)$$

$= -48+70-22 = 0 \Rightarrow$ lin. dep.
so YES lie in same plane

8.) Vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are collinear iff $\vec{v}_2-\vec{v}_1$ and $\vec{v}_3-\vec{v}_2$ are linearly dependent (multiples)



a.) $\vec{v}_2-\vec{v}_1 = \overrightarrow{(3,-6,-9)}$
and
$\vec{v}_3-\vec{v}_2 = \overrightarrow{(-5,10,6)}$
(NOT multiples)

so $\vec{v_1}, \vec{v_2}, \vec{v_3}$ NOT collinear

c.) $\vec{v_2} - \vec{v_1} = \overrightarrow{(-2,-3,-4)}$ and

$\vec{v_3} - \vec{v_2} = \overrightarrow{(-4,-6,-8)}$, where

$\overrightarrow{(-4,-6,-8)} = 2\,\overrightarrow{(-2,-3,-4)}$ (multiples)

so $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are collinear

9.) a.) $a\,\overrightarrow{(0,3,1,-1)} + b\,\overrightarrow{(6,0,5,1)}$
$+ c\,\overrightarrow{(4,-7,1,3)} = \overrightarrow{(0,0,0,0)} \Rightarrow$

$$\begin{cases} 6b + 4c = 0 \\ 3a - 7c = 0 \\ a + 5b + c = 0 \\ -a + b + 3c = 0 \end{cases} \Rightarrow \left.\begin{matrix} 3a - 7c = 0 \\ 6b + 4c = 0 \end{matrix}\right\}$$

so let $c = 3 \Rightarrow 3a = 21 \Rightarrow a = 7$
and $6b = -12 \Rightarrow b = -2$, so

$(7)\overrightarrow{(0,3,1,-1)} + (-2)\overrightarrow{(6,0,5,1)} + (3)\overrightarrow{(4,-7,1,3)} = \overrightarrow{(0,0,0,0)}$
$\Rightarrow$ lin. dep.

OR

$$\begin{bmatrix} 0 & 3 & 1 & -1 \\ 6 & 0 & 5 & 1 \\ 4 & -7 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & 1 & -1 \\ 12 & 0 & 10 & 2 \\ -12 & 21 & -3 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 3 & 1 & -1 \\ 12 & 0 & 10 & 2 \\ 0 & 21 & 7 & -7 \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & 1 & -1 \\ 12 & 0 & 10 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $\overrightarrow{(4,-7,1,3)}$ is a linear combination of $\overrightarrow{(0,3,1,-1)}$ and $\overrightarrow{(6,0,5,1)}$ $\Rightarrow$ lin. dep.

10.) b.) $a\,\overrightarrow{(1,2,3,4)} + b\,\overrightarrow{(0,1,0,-1)} = \overrightarrow{(1,3,3,3)} \Rightarrow$

$$\begin{cases} a = 1 \\ 2a + b = 3 \Rightarrow 2(1) + b = 3 \Rightarrow b = 1, \text{ so} \\ 3a = 3 \\ 4a - b = 3 \end{cases}$$

i.) $\overrightarrow{(1,3,3,3)} = (1)\,\overrightarrow{(1,2,3,4)} + (1)\,\overrightarrow{(0,1,0,-1)}$

and ii.) $\overrightarrow{(1,2,3,4)} = (1)\,\overrightarrow{(1,3,3,3)} + (-1)\,\overrightarrow{(0,1,0,-1)}$

and iii) $\overrightarrow{(0,1,0,-1)} = (1)\,\overrightarrow{(1,3,3,3)} + (-1)\,\overrightarrow{(1,2,3,4)}$

11.) $\det \begin{bmatrix} \lambda & -1/2 & -1/2 \\ -1/2 & \lambda & -1/2 \\ -1/2 & -1/2 & \lambda \end{bmatrix} = 0 \Rightarrow$

$$\lambda\left(\lambda^2 - \frac{1}{4}\right) + \frac{1}{2}\left(-\frac{1}{2}\lambda - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{4} + \frac{1}{2}\lambda\right)$$

$$= \lambda^3 - \frac{1}{4}\lambda - \frac{1}{4}\lambda - \frac{1}{8} - \frac{1}{8} - \frac{1}{4}\lambda$$

$= \lambda^3 - \frac{3}{4}\lambda - \frac{1}{4} = 0 \Rightarrow$

$4\lambda^3 - 3\lambda - 1 = 0 \Rightarrow$

$(\lambda - 1)(4\lambda^2 + 4\lambda + 1) = 0 \Rightarrow$

$(\lambda - 1)(2\lambda + 1)^2 = 0 \Rightarrow$

$$
\begin{array}{r|l}
 & 4\lambda^2 + 4\lambda + 1 \\
\hline
\lambda - 1 & 4\lambda^3 - 3\lambda - 1 \\
 & -(4\lambda^3 - 4\lambda^2) \\
\cline{2-2}
 & 4\lambda^2 - 3\lambda - 1 \\
 & -(4\lambda^2 - 4\lambda) \\
\cline{2-2}
 & \lambda - 1
\end{array}
$$

if $\lambda = 1$ or $\lambda = -1/2$, then determinant $= 0$ and vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly dependent

12.) Vector $\vec{v}$ forms a linearly dependent set iff $\vec{v} = \vec{0}$, since $1\vec{v} = 1\vec{0} = \vec{0}$, but $1 \neq 0$.

13.) a.) $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$T(\overrightarrow{a,b}) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - b \\ 2b \end{bmatrix}$; then

$T(\overrightarrow{1,2}) = \begin{bmatrix} 1 - 2 \\ 2(2) \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$,

$T(\overrightarrow{-1,1}) = \begin{bmatrix} -1 - 1 \\ 2(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, and

$\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are linearly independent since they are not multiples of each other.

b.) $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$T(\overrightarrow{a,b}) = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a-b \\ -2a+2b \end{bmatrix}$; then

$T(\overrightarrow{1,2}) = \begin{bmatrix} 1-2 \\ -2+4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$,

$T(-1,1) = \begin{bmatrix} -1-1 \\ 2+2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and

$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$ are linearly dependent since they are multiples of each other

15.) a.) linearly independent since vectors cannot form a triangle

b.) linearly dependent since vectors can form a triangle

16.) a.)

$(-1)(6) + (2)(3\sin^2 x) + (3)(2\cos^2 x)$

$= -6 + 6\sin^2 x + 6\cos^2 x$

$= -6 + 6(\sin^2 x + \cos^2 x)$

$= -6 + 6(1) = 0$, so

6, $3\sin^2 x$, and $2\cos^2 x$ are linearly dependent

b.) $W(x) = \begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix}$

$= -x\sin x - \cos x \neq 0$, so x and $\cos x$ are linearly independent

c.) $W(x) = \begin{vmatrix} 1 & \sin x & \sin 2x \\ 0 & \cos x & 2\cos 2x \\ 0 & -\sin x & -4\sin 2x \end{vmatrix}$

$= 1 \cdot (-4\cos x \sin 2x + 2\cos 2x \sin x)$
$- \sin x \cdot (0 - 0) + \sin 2x \cdot (0 - 0)$

$= -4\cos x \cdot (2\sin x \cos x)$
$+ 2(2\cos^2 x - 1)\sin x$

$= -8\cos^2 x \sin x + 4\cos^2 x \sin x$
$- 2\sin x$

$= -4\cos^2 x \sin x - 2\sin x$

$= -2\sin x\,(2\cos^2 x + 1) \neq 0,$

so 1, $\sin x$, and $\sin 2x$ are linearly independent

e.) $(5)(3-x)^2 + (-5)(x^2-6x) + (-9)(5)$

$= 5(9-6x+x^2) - 5(x^2-6x) - 45$

$= 45 - 30x + 5x^2 - 5x^2 + 30x - 45$

$= 0$, so $(3-x)^2$, (x^2-6x), and 5 are linearly dependent

18.) $W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$

$= -\sin^2 x - \cos^2 x = -(\sin^2 x + \cos^2 x)$

$= -1 \neq 0$, so $\sin x$ and $\cos x$ are linearly independent

19.) a.) $W(x) = \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} = (1)(1)(e^x) = e^x \neq 0,$

so 1, x, and e^x are linearly independent b.) $$W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = (1)(1)(2) = 2 \neq 0,$$

so 1, x, and x^2 are linearly independent

20.) $$W(x) = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & (x+1)e^x & (x^2+2x)e^x \\ e^x & (x+2)e^x & (x^2+4x+2)e^x \end{vmatrix}$$

$$= e^x\left[(x+1)e^x(x^2+4x+2)e^x - (x^2+2x)e^x(x+2)e^x\right]$$
$$- xe^x\left[e^x(x^2+4x+2)e^x - (x^2+2x)e^x(x+2)e^x\right]$$
$$+ x^2e^x\left[e^x(x+2)e^x - (x+1)e^xe^x\right]$$

21.) $$W(x) = \begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ -\sin x & -\cos x & -2\sin x - x\cos x \end{vmatrix}$$

$$= \sin x\left((2\sin^2 x + x\sin x\cos x) - (-\cos^2 x + x\sin x\cos x)\right)$$

$$- \cos x\left(\left(-2\sin x\cos x - x\cos^2 x\right)\right.$$
$$\left.- \left(-\sin x\cos x + x\sin^2 x\right)\right)$$
$$+ x\cos x\left(-\cos^2 x - \sin^2 x\right)$$

$$= \sin x\left(2\sin^2 x + \cos^2 x - 2x\sin x\cos x\right)$$
$$- \cos x\left(-\sin x\cos x - x\cos^2 x - x\sin^2 x\right)$$
$$+ x\cos x\,(-1)\left(\cos^2 x + \sin^2 x\right)$$

$$= \sin x\left(\sin^2 x + \left(\sin^2 x + \cos^2 x\right) - 2x\sin x\cos x\right)$$
$$- \cos x\left(-\sin x\cos x - x\left(\cos^2 x + \sin^2 x\right)\right)$$
$$- x\cos x \cdot (1)$$
$$= \sin x\left(\sin^2 x + 1 - 2x\sin x\cos x\right)$$
$$- \cos x\left(-\sin x\cos x - x\,(1)\right)$$
$$- x\cos x$$

$$= \sin^3 x + \sin x - 2x\sin^2 x\cos x$$
$$+ \sin x\cos^2 x + x\cancel{\cos x} - x\cancel{\cos x}$$
$$= \sin x\left(\sin^2 x + 1 - 2x\sin x\cos x + \cos^2 x\right)$$
$$= 1$$

$$= \sin x\left(1 + 1 - 2x\sin x\cos x\right)$$

$= \sin x\,(2 - 2x \sin x \cos x)$

$= 2 \sin x\,(1 - x \sin x \cos x) \neq 0,$

so $\sin x$, $\cos x$, and $x \cos x$ are linearly independent

22.) $(1)(\vec{u}-\vec{v}) + (1)(\vec{v}-\vec{w}) + (1)(\vec{w}-\vec{u})$

$= \cancel{\vec{u}} - \cancel{\vec{v}} + \cancel{\vec{v}} - \cancel{\vec{w}} + \cancel{\vec{w}} - \cancel{\vec{u}} = \vec{0}$, so

$\vec{u}-\vec{v}$, $\vec{v}-\vec{w}$, and $\vec{w}-\vec{u}$ are linearly dependent

24.) assume $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are lin. ind.:

$(*)$ $\begin{cases} \text{If } k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0}, \text{ then} \\ k_1 = 0,\ k_2 = 0, \text{ and } k_3 = 0 \end{cases}$

a.) Show $\{\vec{v}_2, \vec{v}_3\}$ are lin. ind.:

$\alpha\vec{v}_2 + \beta\vec{v}_3 = \vec{0} \Longrightarrow (0)\vec{v}_1 + \alpha\vec{v}_2 + \beta\vec{v}_3 = \vec{0}$

$\Rightarrow 0 = 0$, $\alpha = 0$, and $\beta = 0$ (by $(*)$), so $\vec{v}_2$ and $\vec{v}_3$ are lin. ind.

b.) Show $\{\vec{v}_1\}$ is lin. ind.:

$\alpha\vec{v}_1 = \vec{0} \Rightarrow \alpha\vec{v}_1 + (0)\vec{v}_2 + (0)\vec{v}_3 = \vec{0}$

$\Rightarrow \alpha = 0$, $0 = 0$, and $0 = 0$ (by (*)), so $\{\vec{v}_1\}$ is lin. ind.

26.) $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a lin. dep. set so $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0}$ where $k_1, k_2,$ and k_3 are not all zero; assume $\vec{v}_4 \in V$ but $\vec{v}_4 \notin S$. Show $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a lin. dep. set:

$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 + (0)\vec{v}_4 = \vec{0}$, but $k_1, k_2,$ and k_3 are not all zero, so this set is lin. dep.

28.) Prove that in $P_1 = \{a + bx \mid a, b \in \mathbb{R}\}$ every set of 3 vectors is linearly dependent:

assume $p_1(x), p_2(x), p_3(x) \in P_1$. Then

$p_1(x) = \alpha_1 + \alpha_2 x$, $p_2(x) = \beta_1 + \beta_2 x$, and $p_3(x) = \gamma_1 + \gamma_2 x$. Find solutions k_1, k_2, k_3 not all zero to equation

$k_1 p_1(x) + k_2 p_2(x) + k_3 p_3(x) = 0 \Rightarrow$

$k_1(\alpha_1 + \alpha_2 x) + k_2(\beta_1 + \beta_2 x) + k_3(\gamma_1 + \gamma_2 x) = 0 \Rightarrow$

$k_1\alpha_1 + k_1\alpha_2 x + k_2\beta_1 + k_2\beta_2 x + k_3\gamma_1 + k_3\gamma_2 x = 0 \Rightarrow$

$(\alpha_1 k_1 + \beta_1 k_2 + \gamma_1 k_3)(1)$

$+ (\alpha_2 k_1 + \beta_2 k_2 + \gamma_2 k_3)(x) = 0 \Rightarrow$

$\begin{cases} \alpha_1 k_1 + \beta_1 k_2 + \gamma_1 k_3 = 0 \\ \alpha_2 k_1 + \beta_2 k_2 + \gamma_2 k_3 = 0 \end{cases}$ (since $\{1, x\}$ is a linearly independent set: $W(x) = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$);

solve the system of equations for k_1, k_2, k_3:

$\left[\begin{array}{ccc|c} \alpha_1 & \beta_1 & \gamma_1 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 \end{array}\right]$ is row-equivalent

to $\left[\begin{array}{ccc|c} 1 & 0 & \alpha & 0 \\ 0 & 1 & \beta & 0 \end{array}\right]$ or $\left[\begin{array}{ccc|c} 1 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ or $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

or $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ or $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$; all of these

have <u>nonzero</u> solutions for

k_1, k_2, and k_3, so $p_1(x), p_2(x)$, and $p_3(x)$ are linearly dependent

29.) Assume that set $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and assume $\vec{v}_3 \notin \text{span}\{\vec{v}_1, \vec{v}_2\} = \{\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 \mid \text{for all } \alpha_1, \alpha_2 \in \mathbb{R}\}$.

Show that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent:

Let $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$.

<u>Case 1</u>: If $c_3 = 0$, then

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2 + 0\,\vec{v}_3$$

$$= c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0} \Rightarrow c_1 = 0 \text{ or } c_2 = 0$$

since $\vec{v}_1, \vec{v}_2$ are lin. ind.

<u>Case 2</u>: If $c_3 \neq 0$, then

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \Rightarrow$$

$$c_3\vec{v}_3 = {}^-c_1\vec{v}_1 + {}^-c_2\vec{v}_2 \Rightarrow$$

$$\vec{v}_3 = \left(\frac{-c_1}{c_3}\right)\vec{v}_1 + \left(\frac{-c_2}{c_3}\right)\vec{v}_2,$$ which

is impossible since $\vec{v}_3 \notin \text{span}\{\vec{v}_1, \vec{v}_2\}$.

Since Case 1.) is the only possibility, then $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly independent.

TRUE/FALSE

(a) F (b) T (c) F (d) T

(e) T (f) F (g) T (h) F