Section 4.4

1.) $\det\begin{bmatrix}2 & 1\\ 3 & 0\end{bmatrix} = 0-3 = -3 \neq 0$, so

$\overrightarrow{(2,1)}$ and $\overrightarrow{(3,0)}$ form a basis for $\mathbb{R}^2$

2.) $\det\begin{bmatrix}3 & 1 & -4\\ 2 & 5 & 6\\ 1 & 4 & 8\end{bmatrix} = 3(40-24)-1(16-6)$
$-4(8-5)$

$= 48-10-12 = 26 \neq 0$, so

$\overrightarrow{(3,1,-4)}$, $\overrightarrow{(2,5,6)}$, and $\overrightarrow{(1,4,8)}$ form a basis for $\mathbb{R}^3$

3.) Let $p_1(x) = x^2+1$, $p_2(x) = x^2-1$, and $p_3(x) = 2x-1$ be in P_3. Show $\{p_1(x), p_2(x), p_3(x)\}$ forms a basis for P_3:

(SPAN): Let $a+bx+cx^2$ be any polynomial in P_3, and assume

$$a+bx+cx^2 = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x)$$

$$= \alpha_1(x^2+1) + \alpha_2(x^2-1) + \alpha_3(2x-1)$$

$= \alpha_1 x^2 + \alpha_1 + \alpha_2 x^2 - \alpha_2 + 2\alpha_3 x - \alpha_3$

$= (\alpha_1 - \alpha_2 - \alpha_3) + (2\alpha_3)x + (\alpha_1 + \alpha_2)x^2 \Rightarrow$

$(\alpha_1 - \alpha_2 - \alpha_3 - a)(1) + (2\alpha_3 - b)(x) + (\alpha_1 + \alpha_2 - c)(x^2) = 0$

$\Rightarrow$

$$\begin{cases} \alpha_1 - \alpha_2 - \alpha_3 - a = 0 \\ 2\alpha_3 - b = 0 \\ \alpha_1 + \alpha_2 - c = 0 \end{cases}$$

(since $1, x, x^2$ are linearly independent:

$W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = (1)(1)(2) = 2 \neq 0$)

$\Rightarrow$

$$\begin{cases} \alpha_1 - \alpha_2 - \alpha_3 = a \\ 2\alpha_3 = b \\ \alpha_1 + \alpha_2 = c \end{cases} \Rightarrow$$

$$\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 0 & 0 & 2 & b \\ 1 & 1 & 0 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & a + b/2 \\ 0 & 0 & 1 & b/2 \\ 0 & 2 & 1 & c - a \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & a + b/2 \\ 0 & 0 & 1 & b/2 \\ 0 & 2 & 0 & c - a - b/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & a + b/2 \\ 0 & 0 & 1 & b/2 \\ 0 & 1 & 0 & c/2 - a/2 - b/4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & a/2 + b/4 + c/2 \\ 0 & 0 & 1 & b/2 \\ 0 & 1 & 0 & c/2 - a/2 - b/4 \end{array}\right] \Rightarrow$$

$\alpha_1 = \frac{a}{2} + \frac{b}{4} + \frac{c}{2}$, $\alpha_2 = \frac{c}{2} - \frac{a}{2} - \frac{b}{4}$, and $\alpha_3 = b/2$, so $\{P_1(x), P_2(x), P_3(x)\}$ spans P_2.

(LINEAR INDEPENDENCE):

$$W(x) = \begin{vmatrix} x^2+1 & x^2-1 & 2x-1 \\ 2x & 2x & 2 \\ 2 & 2 & 0 \end{vmatrix}$$

$= (x^2+1)(0-4) - (x^2-1)(0-4) + (2x-1)(\overset{0}{\cancel{4x-4x}})$

$= -4x^2 - 4 + 4x^2 - 4 = -8 = 0$, so set $\{P_1(x), P_2(x), P_3(x)\}$ is linearly independent; thus $\{P_1(x), P_2(x), P_3(x)\}$ forms basis for P_2.

6.) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any matrix in M_{22} and assume that

$$(*)\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_4 & \alpha_1 - \alpha_2 - \alpha_3 \\ \alpha_1 + \alpha_3 & \alpha_1 \end{bmatrix} \Rightarrow$$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_4 = a \\ \alpha_1 - \alpha_2 - \alpha_3 = b \\ \alpha_1 + \alpha_3 = c \\ \alpha_1 = d \end{cases} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & a \\ 1 & -1 & -1 & 0 & b \\ 1 & 0 & 1 & 0 & c \\ 1 & 0 & 0 & 0 & d \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & a-d \\ 0 & -1 & -1 & 0 & b-d \\ 0 & 0 & 1 & 0 & c-d \\ 1 & 0 & 0 & 0 & d \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & a-d \\ 0 & 0 & -1 & 1 & a+b-2d \\ 0 & 0 & 1 & 0 & c-d \\ 1 & 0 & 0 & 0 & d \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & a-d \\ 0 & 0 & 0 & 1 & a+b+c-3d \\ 0 & 0 & 1 & 0 & c-d \\ 1 & 0 & 0 & 0 & d \end{array}\right]$$

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & -b-c+2d \\ 0 & 0 & 0 & 1 & a+b+c-3d \\ 0 & 0 & 1 & 0 & c-d \\ 1 & 0 & 0 & 0 & d \end{array}\right] \Longrightarrow \end{array}$$

$\alpha_1 = d$, $\alpha_2 = -b-c+2d$,
$\alpha_3 = c-d$, and $\alpha_4 = a+b+c-3d$,
so the 4 matrices $\boxed{\text{SPAN}}$ M_{22};

if $a=b=c=d=0$, then
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$
by (*), so the 4 matrices
are $\boxed{\text{linearly independent}}$;
so the 4 matrices form a basis for M_{22}.

7.) a.) $\det\begin{bmatrix} 2 & -3 & 1 \\ 4 & 1 & 1 \\ 0 & -7 & 1 \end{bmatrix} = 2(1+7)-(-3)(4-0)$
$+1(-28-0)$

$= 16+12-28 = 0$, ~~so~~ $\overrightarrow{(2,-3,1)}$, $\overrightarrow{(4,1,1)}$, and $\overrightarrow{(0,-7,1)}$ are linearly dependent so cannot form a basis for $\mathbb{R}^3$

8.) I.) Let $a+bx+cx^2 \in P_2$ and assume

$$a+bx+cx^2 = \alpha_1(1-3x+2x^2) + \alpha_2(1+x+4x^2) + \alpha_3(1-7x)$$

$$= \alpha_1 - 3\alpha_1 x + 2\alpha_1 x^2 + \alpha_2 + \alpha_2 x + 4\alpha_2 x^2 + \alpha_3 - 7\alpha_3 x$$

$$= (\alpha_1+\alpha_2+\alpha_3) + (-3\alpha_1+\alpha_2-7\alpha_3)x + (2\alpha_1+4\alpha_2)x^2 \Rightarrow$$

$$0 = (\alpha_1+\alpha_2+\alpha_3-a)\cdot 1 + (-3\alpha_1+\alpha_2-7\alpha_3-b)x + (2\alpha_1+4\alpha_2-c)x^2 \Rightarrow$$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 - a = 0 \\ -3\alpha_1 + \alpha_2 - 7\alpha_3 - b = 0 \\ 2\alpha_1 + 4\alpha_2 - c = 0 \end{cases}$$

(Since 1, x, and x^2 are linearly independent : $\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = (1)(1)(2)$

$= W(x) \neq 0$) $\Longrightarrow$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = a \\ -3\alpha_1 + \alpha_2 - 7\alpha_3 = b \\ 2\alpha_1 + 4\alpha_2 = c \end{cases} \Longrightarrow$$

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ -3 & 1 & -7 & b \\ 2 & 4 & 0 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 4 & -4 & 3a+b \\ 0 & 2 & -2 & -2a+c \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 0 & 0 & 7a+b-2c \\ 0 & 2 & -2 & -2a+c \end{array}\right],$$ so system is

solvable iff $\boxed{7a+b-2c=0}$;

but if

$a=1$, $b=1$, and $c=1$ then $7(1)+(1)-2(1)$

$= 6 \neq 0$, so

$p(x) = 1 + x + x^2$ is NOT in the SPAN of $1-3x+2x^2$, $1+x+4x^2$ and $1-7x$ $\Rightarrow$ these 3 vectors are not a basis for P_2

$\boxed{\text{OR}}$

II.) ~~assume~~

$$k_1(1-3x+2x^2)+k_2(1+x+4x^2)+k_3(1-7x)=0 \Rightarrow$$

$$(k_1+k_2+k_3)1+(-3k_1+k_2-7k_3)x + (2k_1+4k_2)x^2 = 0 \Rightarrow$$

$$\begin{cases} k_1+k_2+k_3=0 \\ -3k_1+k_2-7k_3=0 \\ 2k_1+4k_2=0 \end{cases}$$ since $1, x,$ and x^2 are lin. ind

$$\Rightarrow \begin{array}{c} \begin{matrix} k_1 & k_2 & k_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -3 & 1 & -7 & 0 \\ 2 & 4 & 0 & 0 \end{array}\right] \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow \begin{cases} k_1+2k_3=0 \\ k_2-k_3=0 \end{cases}$$

$\Rightarrow k_3 = 1,\ k_2 = 1,$ and $k_1 = -2$

is a NONZERO solution, so $1-3x+2x^2$, $1+x+4x^2$, and $1-7x$ are LINERLY DEPENDENT, so cannot be a basis for P_2

9.) Show $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ are linearly dependent: assume

$$k_1 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix} + k_3 \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} k_1 + 2k_2 + k_3 = 0 \\ -2k_2 - k_3 - k_4 = 0 \\ k_1 + 3k_2 + k_3 + k_4 = 0 \\ k_1 + 2k_2 + k_4 = 0 \end{cases} \Rightarrow$$

$$\begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 0 \\ 1 & 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \end{array} \Rightarrow$$

$k_1 - k_4 = 0$, $k_2 + k_4 = 0$, and $k_3 - k_4 = 0$; if $k_4 = 1$, then $k_1 = 1$, $k_2 = -1$, and $k_3 = 1$, so the 4 2×2 matrices are linearly dependent.

10.) a.) $\cos 2x = 2\cos^2 x - 1$ (TRIG identity)

$= 2\cos^2 x - (\cos^2 x + \sin^2 x)$ (TRIG identity)

$= \cos^2 x - \sin^2 x \Rightarrow$

$(1)\cos 2x + (-1)\cos^2 x + (1)\sin^2 x = 0$,

so $\cos 2x$, $\cos^2 x$, and $\sin^2 x$ are linearly <u>dependent</u>, so they cannot form a basis

b.) $\{1, \cos 2x\}$ is a basis for

$V = \text{span}\{\cos 2x, \cos^2 x, \sin^2 x\}$:

I.) $W(x) = \begin{vmatrix} 1 & \cos 2x \\ 0 & -2\sin 2x \end{vmatrix}$

$= -2\sin 2x \neq 0$, so 1 and $\cos 2x$ are linearly independent

II.) $\cos 2x = (1)\cos 2x + (0)1$;

$\cos^2 x = (2)\cos 2x - (1)1$; and

$$\sin^2 x = 1 - \cos^2 x$$
$$= 1 - [(2)\cos 2x - (1)1]$$
$$= (-2)\cos 2x + (2)1;$$ so

$\operatorname{span}\{1, \cos 2x\} = V$; thus,

$\{1, \cos x\}$ is a basis for V

14.) a.) $p(x) = 4 - 3x + x^2$
$$= (4)1 + (-3)x + (1)x^2,$$ so

$$(p(x))_S = \overrightarrow{(4, -3, 1)}$$

b.) $p(x) = 2 - x + x^2 = k_1(1+x)$
$$+ k_2(1+x^2) + k_3(x+x^2)$$
$$= (k_1+k_2) + (k_1+k_3)x + (k_2+k_3)x^2 \Rightarrow$$

$$0 = \underbrace{(k_1+k_2-2)}_{0}(1) + \underbrace{(k_1+k_3+1)}_{0}(x) + \underbrace{(k_2+k_3-1)}_{0}(x^2)$$

$$\Rightarrow \begin{cases} k_1 + k_2 = 2 \\ k_1 + k_3 = -1 \\ k_2 + k_3 = 1 \end{cases}$$ (since $1, x, x^2$ are lin. ind.) $\Rightarrow$

$$\begin{array}{cccc} k_1 & k_2 & k_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \end{array} \Rightarrow$$

$k_1 = 0$, $k_2 = 2$, and $k_3 = -1$, so

$$(p(x))_S = \overrightarrow{(0, 2, -1)}$$

20.) If $\vec{0} \in \{\vec{0}, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = S$, then
$(1)\vec{0} + (0)\vec{v}_1 + (0)\vec{v}_2 + \cdots + (0)\vec{v}_n = \vec{0}$,
so set S is linearly dependent

29.) $R^\infty = \{(x_1, x_2, x_3, \dots) \mid x_i \in \mathbb{R} \text{ for } i = 1, 2, 3, \dots\}$;
the set of vectors
$S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \dots\}$, where

$\vec{v}_1 = (1, 0, 0, 0, \ldots)$,
$\vec{v}_2 = (0, 1, 0, 0, \ldots)$,
$\vec{v}_3 = (0, 0, 1, 0, \ldots)$,
$\vec{v}_4 = (0, 0, 0, 1, 0, \ldots)$
$\vdots$
forms a basis (infinite)
for R^∞

31.) Assume that V is an ∞-dimensional subspace of vector space W, then W is ∞-dimensional:
(This is a PROOF by CONTRADICTION.)

Since V is ∞-dimensional it has no FINITE spanning set or basis. Assume that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots\}$ is a basis for V.
Show that W is ∞-dimensional.
Assume that W is finite-dimensional with basis $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$, i.e.,

$\text{span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\} = W$. But $V \subseteq W$ and $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots\} = V$; then

$\vec{v}_i \in \text{span}\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k\}$ for $i = 1, 2, 3, \ldots$

$$\Rightarrow V = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots\}$$
$$\subseteq \text{span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}.$$

This implies that V can be spanned by a FINITE # of vectors $\Rightarrow$ V is FINITE-dimensional. This is a CONTRADICTION. Thus, W is ∞-dimensional.

TRUE / FALSE

(a) F (b) F (c) T (d) T (e) F