Section 4.5

1.) $\begin{cases} x_1 + x_2 - x_3 = 0 \\ -2x_1 - x_2 + 2x_3 = 0 \\ -x_1 + x_3 = 0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$

$x_1 - x_3 = 0$ and $x_2 = 0$, so let $x_3 = t$ any #

$\Rightarrow x_1 = t$ so solution

$\overrightarrow{(x_1, x_2, x_3)} = \overrightarrow{(t, 0, t)} = t\,\overrightarrow{(1,0,1)}$, so

$\{\overrightarrow{(1,0,1)}\}$ is basis for the solution space, which has dimension 1

2.) $\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ -1 & -3 & -1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & -8 & -2 & -8 & 0 \\ 1 & 3 & 1 & 3 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 0 & 1 & 1/4 & 1 & 0 \\ 1 & 3 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 1 & 1/4 & 1 & 0 \\ 1 & 0 & 1/4 & 0 & 0 \end{array}\right] \Rightarrow$

$x_1 + \frac{1}{4}x_3 = 0$ and $x_2 + \frac{1}{4}x_3 + x_4 = 0$, so let $x_4 = t$ any #, $x_3 = s$ any # $\Rightarrow$

$x_1 = \frac{-1}{4}s$ and $x_2 = \frac{-1}{4}s - t$ $\Rightarrow$

$\overrightarrow{(x_1, x_2, x_3, x_4)} = \overrightarrow{\left(\frac{-1}{4}s, \frac{-1}{4}s - t, s, t\right)}$

$= \overrightarrow{\left(\frac{-1}{4}s, \frac{-1}{4}s, s, 0\right)} + \overrightarrow{(0, -t, 0, t)}$

$= \frac{-1}{4}s\,\overrightarrow{(1, 1, -4, 0)} + t\,\overrightarrow{(0, -1, 0, 1)}$, so

$\{\overrightarrow{(1, 1, -4, 0)}, \overrightarrow{(0, -1, 0, 1)}\}$ is a basis for the solution space, which has dimension 2

3.) $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 5x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 0 & 1 & -7 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & -7 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 0 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\Rightarrow x_1 = 0, x_2 = 0, x_3 = 0$ so solution space is $\{\overrightarrow{(0, 0, 0)}\}$ and its dimension is zero

6.) $\begin{cases} x+y+z=0 \\ 3x+2y-2z=0 \\ 4x+3y-z=0 \\ 6x+5y+z=0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$

$x-4z=0$ and $y+5z=0$, so let $z=t$ any # $\Rightarrow x=4t$, $y=-5t \Rightarrow$ $\overrightarrow{(x,y,z)} = \overrightarrow{(4t,-5t,t)} = t\,\overrightarrow{(4,-5,1)}$, so $\{\overrightarrow{(4,-5,1)}\}$ is a basis for the solution space, which has dimension 1

7.) a.) $3x-2y+5z=0 \Rightarrow$ let $z=t$ any #, $y=s$ any # $\Rightarrow 3x-2s+5t=0 \Rightarrow$ $x=\frac{2}{3}s-\frac{5}{3}t \Rightarrow \overrightarrow{(x,y,z)} = \overrightarrow{(\frac{2}{3}s-\frac{5}{3}t,s,t)}$

$= \overrightarrow{(\frac{2}{3}s,s,0)} + \overrightarrow{(-\frac{5}{3}t,0,t)}$

$= \frac{1}{3}s\,\overrightarrow{(2,3,0)} + \frac{1}{3}t\,\overrightarrow{(-5,0,3)}$, so

$\{\overrightarrow{(2,3,0)}, \overrightarrow{(-5,0,3)}\}$ is a basis for the subspace, which has dimension 2

b.) $x - y = 0$, so let $z = t$ any #, $y = s$ any # $\Rightarrow$ $x = s$ $\Rightarrow$

$\overrightarrow{(x,y,z)} = \overrightarrow{(s,s,t)} = \overrightarrow{(s,s,0)} + \overrightarrow{(0,0,t)}$

$= s\,\overrightarrow{(1,1,0)} + t\,\overrightarrow{(0,0,1)}$, so

$\{\overrightarrow{(1,1,0)}, \overrightarrow{(0,0,1)}\}$ is a basis for the subspace, which has dimension 2

c.) $\overrightarrow{(x,y,z)} = \overrightarrow{(2t,-t,4t)}$

$= t\,\overrightarrow{(2,-1,4)}$, so $\{\overrightarrow{(2,-1,4)}\}$ is a basis for the subspace, which has dimension 1

8.) a.) $\overrightarrow{(a,b,c,0)} = \overrightarrow{(a,0,0,0)} + \overrightarrow{(0,b,0,0)}$

$+ \overrightarrow{(0,0,c,0)}$

$= a\,\overrightarrow{(1,0,0,0)} + b\,\overrightarrow{(0,1,0,0)} + c\,\overrightarrow{(0,0,1,0)}$,

so $\{\overrightarrow{(1,0,0,0)}, \overrightarrow{(0,1,0,0)}, \overrightarrow{(0,0,1,0)}\}$

is a basis for the subspace,
which has dimension 3

b.) $\overrightarrow{(a,b,c,d)} = \overrightarrow{(a,b,a-b,a+b)}$

$= \overrightarrow{(a,0,a,a)} + \overrightarrow{(0,b,-b,b)}$

$= a\,\overrightarrow{(1,0,1,1)} + b\,\overrightarrow{(0,1,-1,1)}$, so

$\{\overrightarrow{(1,0,1,1)}, \overrightarrow{(0,1,-1,1)}\}$ is a basis for

the subspace, which has
dimension 2

9.) a.) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$+ c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cdots + h\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ dimension $= (3)(3) = 9$

b.) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$+ c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$+ f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$ dimension $= 6$

10.) $a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$= 0 + a_1 x + a_2 x^2 + a_3 x^3$

$= a_1(x) + a_2(x^2) + a_3(x^3) \Rightarrow$

dimension $= 3$

12.) a.) $\begin{bmatrix} -1 & 2 & 3 \\ 1 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ add $\vec{v}_3 = \overrightarrow{(0,1,0)}$, so

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$

b.) $\begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & -2 \end{bmatrix} \Rightarrow$ add $\vec{v}_3 = \overrightarrow{(0,0,1)}$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$

13.) $\begin{bmatrix} 1 & -4 & 2 & -3 \\ -3 & 8 & -4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 & -3 \\ 0 & -4 & 2 & -3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 2 & -3 \end{bmatrix} \Rightarrow$ add $\vec{v}_3 = \overrightarrow{(0,0,1,0)}$ and $\vec{v}_4 = \overrightarrow{(0,0,0,1)}$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$

14.) Assume $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for. Show that $\vec{u}_1 = \vec{v}_1$, $\vec{u}_2 = \vec{v}_1 + \vec{v}_2$, and $\vec{u}_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$:

(Apply Theorem 4.5.4) It suffices to show that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ are linearly independent: Let

$$\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \alpha_3 \vec{u}_3 = \vec{0} \Rightarrow$$

$$\alpha_1(\vec{v}_1) + \alpha_2(\vec{v}_1 + \vec{v}_2) + \alpha_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{0} \Rightarrow$$

$$(\alpha_1 + \alpha_2 + \alpha_3)\vec{v}_1 + (\alpha_2 + \alpha_3)\vec{v}_2 + (\alpha_3)\vec{v}_3 = \vec{0} \Rightarrow$$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_2 + \alpha_3 = 0 \\ \alpha_3 = 0 \end{cases}$$ (since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are lin. ind.)

$\Rightarrow \alpha_2 + \alpha_3 = \alpha_2 + 0 = \alpha_2 = 0$ and

$\alpha_1 + \alpha_2 + \alpha_3 = \alpha_1 + 0 + 0 = \alpha_1 = 0 \Rightarrow$

$\vec{u}_1, \vec{u}_2, \vec{u}_3$ are lin. ind.

16.) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow$ add

$\vec{v}_3 = \overrightarrow{(0,0,1,0)}$ and $\vec{v}_4 = \overrightarrow{(0,0,0,1)}$, so

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$

17.) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$

$\{\overrightarrow{(1,0,0)}, \overrightarrow{(0,0,1)}\}$ is a basis for

$\text{span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$

18.) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 3 & 3 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \{\overrightarrow{(1,1,1,0)}, \overrightarrow{(0,0,0,1)}\}$ is a basis for

$\text{span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$

20.) a.) Find all vectors $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$

for which

$$T_A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} :$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ -1 & 4 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 4 & 2 & -1 & 0 \end{array}\right]$$

$$\begin{array}{c} \quad x \quad\ y \quad\ z \quad\ w \\ \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1/2 & -1/4 & 0 \end{array}\right] \end{array} \Rightarrow \begin{cases} x + 2z - w = 0 \\ y + \frac{1}{2}z - \frac{1}{4}w = 0, \end{cases}$$

so let $w = t$ any #, $z = s$ any #

$\Rightarrow x = t - 2s, \quad y = \frac{1}{4}t - \frac{1}{2}s \Rightarrow$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} t-2s \\ \frac{1}{4}t - \frac{1}{2}s \\ s \\ t \end{bmatrix} = \begin{bmatrix} t \\ \frac{1}{4}t \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} -2s \\ -\frac{1}{2}s \\ s \\ 0 \end{bmatrix}$$

$$= \frac{1}{4}t \begin{bmatrix} 4 \\ 1 \\ 0 \\ 4 \end{bmatrix} + \frac{1}{2}s \begin{bmatrix} -4 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \text{ so}$$

$\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is basis for solution space, which has dimension 2

21.) a.) Let $n = 3$, then $\{1, x, x^2, x^3\}$ is a linearly independent set of 4 vectors in $F(-\infty, \infty)$ since the Wronskian is

$$W(x) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = (1)(1)(2)(6) = 12 \neq 0$$

25.) Assume V is a vector space of dimension n and W is a subspace of W. Show that W is finite - dimensional: (PROOF by CONTRADICTION)

Assume that W is NOT finite-dimensional subspace, i.e., W is infinite-dimensional.

Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{v}_{n+1}, \ldots\}$ be a basis for $W \subseteq V$. Then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{v}_{n+1}, \ldots\}$ is a set of more than n vectors in an n-dimensional vector space. By Theorem 4.5.2 (a), this set is LINEARLY DEPENDENT, which contradicts the fact that B is a linearly independent (basis) set. Thus, W is finite-dimensional.

27.) a.) Let $-1+x-2x^2$, $3+3x+6x^2$, and 9 be represented as:

$(-1,1,-2)$, $(3,3,6)$, and $(9,0,0)$, then

$$\begin{bmatrix} -1 & 1 & -2 \\ 3 & 3 & 6 \\ 9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & -2 \\ 0 & 6 & 0 \\ 0 & 9 & -18 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, or

$\{1, x, x^2\}$ is a basis for the subspace.

c.) Let $1+x-3x^2$, $2+2x-6x^2$, and $3+3x-9x^2$ be represented as:

$(1,1,-3)$, $(2,2,-6)$, and $(3,3,-9)$, then

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & 2 & -6 \\ 3 & 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ so $(1,1,-3)$ or

$\{1+x-3x^2\}$ is a basis for the subspace.

TRUE/FALSE

(a) T (b) T (c.) F (d.) T

(e) T (f) T (g) T (h) T

(i) T (j) F (k) F