Section 4.7

1.) a.) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + (2)\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

2.) b.) $\begin{bmatrix} 2 & 1 & 5 \\ 6 & 3 & -8 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix} = (3)\begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$+ (0)\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (-5)\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

3.) a.) $\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 4 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right] \leftarrow$ so $A\vec{x} = \vec{b}$ is NOT solvable $\Rightarrow$ $\vec{b}$ is NOT in the column space of A

b.) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 12 & -8 & -44 \\ 0 & 2 & 0 & -6 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & -8 & -8 \\ 0 & 1 & 0 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -3 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \Longrightarrow$$

$$(1)\begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix} + (-3)\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

5.) a.) $$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5r-2s \\ s \\ s+t \\ t \end{bmatrix} = \begin{bmatrix} 5r \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2s \\ s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t \\ t \end{bmatrix}$$

$$= r\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

b.) Since $A\begin{bmatrix} 3 \\ 0 \\ -1 \\ 5 \end{bmatrix} = \vec{b}$, the general solution to $A\vec{x} = \vec{b}$ is

$$\vec{x} = r\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -1 \\ 5 \end{bmatrix}$$

7.) a.) $\begin{cases} x_1 - 3x_2 = 1 \\ 2x_1 - 6x_2 = 2 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & -6 & 2 \end{array}\right]$

$\sim \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow x_1 - 3x_2 = 1$, ~~so~~

let $x_2 = t$ any # $\Rightarrow$ solution

to $A\vec{x} = \vec{b}$ is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t+1 \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

and solution to $A\vec{x} = \vec{0}$ is

$$\vec{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

b.) $\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 + x_3 = -2 \\ 2x_1 + x_2 + 3x_3 = 3 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 1 & 0 & 1 & -2 \\ 2 & 1 & 3 & 3 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & -1 & -1 & -7 \\ 0 & -1 & -1 & -7 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow$

$\begin{cases} x_1 + x_3 = -2 \\ x_2 + x_3 = 7 \end{cases} \Rightarrow$ let $x_3 = t$ any # $\Rightarrow$

$x_2 = 7 - t$, $x_1 = -2 - t$, so solution to $A\vec{x} = \vec{b}$ is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2-t \\ 7-t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} + \begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix}$$

$$= t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix}$$; and solution to

$A\vec{x} = \vec{0}$ is $\vec{x} = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

8.) b.) $$\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 4 \\ -2x_1 + x_2 + 2x_3 + x_4 = -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 = 3 \\ 4x_1 - 7x_2 - 5x_4 = -5 \end{cases} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 4 \\ -2 & 1 & 2 & 1 & -1 \\ -1 & 3 & -1 & 2 & 3 \\ 4 & -7 & 0 & -5 & -5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 4 \\ 0 & 5 & -4 & 3 & 7 \\ 0 & 5 & -4 & 3 & 7 \\ 0 & -5 & 4 & -3 & -7 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 4 \\ 0 & 1 & -4/5 & 3/5 & 7/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -7/5 & -1/5 & 6/5 \\ 0 & 1 & -4/5 & 3/5 & 7/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 - \frac{7}{5}x_3 - \frac{1}{5}x_4 = \frac{6}{5} \\ x_2 - \frac{4}{5}x_3 + \frac{3}{5}x_4 = \frac{7}{5} \end{cases}, \text{ so}$$

let $x_4 = t$ any #, $x_3 = s$ any # $\Rightarrow$

$x_2 = \frac{4}{5}s - \frac{3}{5}t + \frac{7}{5}$ and

$x_1 = \frac{7}{5}s + \frac{1}{5}t + \frac{6}{5}$, so solution

~~to~~ $A\vec{x} = \vec{b}$ is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{5}s + \frac{1}{5}t + \frac{6}{5} \\ \frac{4}{5}s - \frac{3}{5}t + \frac{7}{5} \\ s \\ t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{5}s \\ \frac{4}{5}s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{5}t \\ -\frac{3}{5}t \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} \frac{6}{5} \\ \frac{7}{5} \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{5}s \begin{bmatrix} 7 \\ 4 \\ 5 \\ 0 \end{bmatrix} + \frac{1}{5}t \begin{bmatrix} 1 \\ -3 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} \frac{6}{5} \\ \frac{7}{5} \\ 0 \\ 0 \end{bmatrix};$$ and

solution to $A\vec{x} = \vec{0}$ is

$$\vec{x} = \frac{1}{5}s\begin{bmatrix}7\\4\\5\\0\end{bmatrix} + \frac{1}{5}t\begin{bmatrix}1\\-3\\0\\5\end{bmatrix}$$

9.) a.) Solve $A\vec{x} = \vec{0}$ for $\vec{x}$:

$$\left[\begin{array}{ccc|c}1 & -1 & 3 & 0\\5 & -4 & -4 & 0\\7 & -6 & 2 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & -1 & 3 & 0\\0 & 1 & -19 & 0\\0 & 1 & -19 & 0\end{array}\right]$$

$$\sim \begin{array}{c}\begin{array}{ccc}x_1 & x_2 & x_3\end{array}\\ \left[\begin{array}{ccc|c}1 & 0 & -16 & 0\\0 & 1 & -19 & 0\\0 & 0 & 0 & 0\end{array}\right]\end{array} \Rightarrow \begin{cases}x_1 - 16x_3 = 0\\ x_2 - 19x_3 = 0\end{cases}, \text{ so}$$

let $x_3 = t$ any # $\Rightarrow x_2 = 19t$,
$x_1 = 16t$, so

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}16t\\19t\\t\end{bmatrix} = t\begin{bmatrix}16\\19\\1\end{bmatrix},$$ ~~so~~ a

basis for the <u>null space</u> is

$$\left\{\begin{bmatrix}16\\19\\1\end{bmatrix}\right\}$$; now find basis for

row space of A:

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$, so a

basis for <u>row</u> <u>space</u> is

$\{[1\ 0\ -16], [0\ 1\ -19]\}$

10.) a.) Solve $A\vec{x} = \vec{0}$ for $\vec{x}$:

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 7 & 7 & 4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2/7 & 0 \\ 0 & 1 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow$$

$\begin{cases} x_1 + x_3 - \frac{2}{7} x_4 = 0 \\ x_2 + x_3 + \frac{4}{7} x_4 = 0 \end{cases}$, so let $x_4 = t$ any # and

$x_3 = s$ any # $\Rightarrow x_2 = -s - \frac{4}{7}t$,

$x_1 = -s + \frac{2}{7}t$, so

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s+\frac{2}{7}t \\ -s-\frac{4}{7}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s \\ -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 2/7\,t \\ -4/7\,t \\ 0 \\ t \end{bmatrix}$$

$$= s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{7}t\begin{bmatrix} 2 \\ -4 \\ 0 \\ 7 \end{bmatrix},$$ so a basis for the

<u>null space</u> is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 7 \end{bmatrix} \right\}$;

now find a basis for the row space of A :

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 1 & -2/7 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$ so

a basis for the <u>row space</u> is

$$\left\{ [1 \;\; 0 \;\; 1 \;\; -2/7], [0 \;\; 1 \;\; 1 \;\; 4/7] \right\}$$

11.) a.) a basis for the row space is:

$\{[1 \quad 0 \quad 2], [0 \quad 0 \quad 1]\}$; a basis for the column space is: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

b.) a basis for the row space is:

$\{[1 \quad -3 \quad 0 \quad 0], [0 \quad 1 \quad 0 \quad 0]\}$; a basis for the column space is: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

12.) b.) a basis for the row space is:

$\{[1 \quad 2 \quad -1 \quad 5], [0 \quad 1 \quad 4 \quad 3], [0 \quad 0 \quad 1 \quad -7], [0 \quad 0 \quad 0 \quad 1]\}$;

a basis for the column space is: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ -7 \\ 1 \end{bmatrix} \right\}$

13.) a.) $\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ -2 & 5 & -7 & 0 & -6 \\ -1 & 3 & -2 & 1 & -3 \\ -3 & 8 & -9 & 1 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 2 & 6 & 1 & 0 \end{bmatrix}$

$\uparrow \quad \uparrow \quad\quad \uparrow$

$\sim \begin{bmatrix} 1 & 0 & 11 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$ basis for the row space is

$\uparrow \quad \uparrow \quad\quad \uparrow$

(for column space)

$\{[1 \;\; 0 \;\; 11 \;\; 0 \;\; 3], [0 \;\; 1 \;\; 3 \;\; 0 \;\; 0], [0 \;\; 0 \;\; 0 \;\; 1 \;\; 0]\};$

a basis for the column space is: $\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

14.) $\begin{bmatrix} 1 & 1 & -4 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & -1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -4 & -3 \\ 0 & -2 & 10 & 4 \\ 0 & -1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 0 & 8 & -4 \\ 0 & -1 & 1 & 4 \end{bmatrix}$

so basis for subspace is

$\{\overrightarrow{(1,0,-3,1)}, \overrightarrow{(0,0,8,-4)}, \overrightarrow{(0,-1,1,4)}\}$

15.) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 2 & 2 \\ 0 & -3 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & -3 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \end{bmatrix}$ so basis for subspace is

$\{\overrightarrow{(1,0,0,1)}, \overrightarrow{(0,0,1,1)}, \overrightarrow{(0,0,0,2)}, \overrightarrow{(0,-1,0,1)}\}$

16.) $\begin{bmatrix} 1 & 0 & 1 & 1 \\ -3 & 3 & 7 & 1 \\ -1 & 3 & 9 & 3 \\ -5 & 3 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 10 & 4 \\ 0 & 3 & 10 & 4 \\ 0 & 3 & 10 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 10 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

so basis is $\{\overrightarrow{(1,0,1,1)}, \overrightarrow{(-3,3,7,1)}\}$;

$\alpha\overrightarrow{(1,0,1,1)} + \beta\overrightarrow{(-3,3,7,1)} = \overrightarrow{(-1,3,9,3)} \Rightarrow$

$\begin{cases} \alpha - 3\beta = -1 \\ 3\beta = 3 \\ \alpha + 7\beta = 9 \\ \alpha + \beta = 3 \end{cases}$ $\rightarrow \beta = 1$ and $\alpha = 2$, so

$\overrightarrow{(-1,3,9,3)} = (2)\overrightarrow{(1,0,1,1)} + (1)\overrightarrow{(-3,3,7,1)}$;

$$\alpha\,\overrightarrow{(1,0,1,1)} + \beta\,\overrightarrow{(-3,3,7,1)} = \overrightarrow{(-5,3,5,-1)} \Rightarrow$$

$$\begin{cases} \alpha - 3\beta = -5 \\ 3\beta = 3 \\ \alpha + 7\beta = 5 \\ \alpha + \beta = -1 \end{cases} \longrightarrow \beta = 1 \text{ and } \alpha = -2, \text{ so}$$

$$\overrightarrow{(-5,3,5,-1)} = (-2)\,\overrightarrow{(1,0,1,1)} + (1)\,\overrightarrow{(-3,3,7,1)}$$

18.) (SEE problem 10 solution)
Basis for row space is

$$\{[1\ 4\ 5\ 2], [2\ 1\ 3\ 0]\}$$

20.) $\begin{bmatrix} a & b & c & d \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and

$$\begin{bmatrix} a & b & c & d \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow$$

$$\begin{cases} a - b + 3c + 2d = 0 \\ 2a - 2c + 4d = 0 \end{cases} \Longrightarrow$$

$$\begin{cases} a - b + 3c + 2d = 0 \\ a - c + 2d = 0 \end{cases} \Rightarrow$$

$$\begin{cases} a - b + 3c + 2d = 0 \\ b - 4c = 0 \end{cases} \Rightarrow$$

$$\begin{cases} a - c + 2d = 0 \\ b - 4c = 0 \end{cases} \Rightarrow \quad c = t \text{ any \#},$$

$d = s$ any # $\Rightarrow$ $b = 4t$,
$a = t - 2s$; (random choices)

let $t=1$, $s=-1$: $a=3$, $b=4$, $c=1$, $d=-1$
let $t=1$, $s=0$: $a=1$, $b=4$, $c=1$, $d=0$

$$\Rightarrow A = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 4 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{works!}$$

21.) b.) $\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 1 & -1 & 4 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -3 & 4 & 2 \end{array}\right] \sim \begin{array}{cccc} x & y & z & \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 8/3 & 7/3 \\ 0 & 1 & -4/3 & -2/3 \end{array}\right]$$

$$\Rightarrow \begin{cases} x + \frac{8}{3}z = \frac{7}{3} \\ y - \frac{4}{3}z = -\frac{2}{3} \end{cases}, \quad \text{so let } z = t \text{ any \#} \Rightarrow$$

$y = \frac{4}{3}t - \frac{2}{3}$ and $x = \frac{7}{3} - \frac{8}{3}t \Rightarrow$

solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7/3 - 8/3\,t \\ 4/3\,t - 2/3 \\ t \end{bmatrix} = \begin{bmatrix} -8/3\,t \\ 4/3\,t \\ t \end{bmatrix} + \begin{bmatrix} 7/3 \\ -2/3 \\ 0 \end{bmatrix}$$

$$= \frac{1}{3}t \begin{bmatrix} -8 \\ 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 7/3 \\ -2/3 \\ 0 \end{bmatrix}$$

23.) a.) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \text{ and } z = t$$

any #,

so null space is all vectors of the form

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$, i.e., all points on the z-axis;

the column space is the

span of vectors $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\0\end{bmatrix}$, i.e.,

all vectors of the form

$$X\begin{bmatrix}1\\0\\0\end{bmatrix}+Y\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}X\\Y\\0\end{bmatrix}$$, i.e., all points

in the XY-plane

b.) Let $A=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$, then

$$\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}X\\Y\\Z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow \begin{array}{cccc} X & Y & Z & \\ \end{array}\left[\begin{array}{ccc|c}0&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]\Rightarrow$$

$\begin{cases}Y=0\\Z=0\end{cases}$ and $X=t$ any #, so

null space is all vectors of the form

$$\begin{bmatrix}X\\Y\\Z\end{bmatrix}=\begin{bmatrix}t\\0\\0\end{bmatrix}$$, i.e., all points on the X-axis;

the column space is the span of vectors

$\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$, i.e., all vectors of the form

$y\begin{bmatrix}0\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\y\\z\end{bmatrix}$, i.e., all points in the yz-plane

24.) a.) Let $A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, then

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

b.) Let $A=\begin{bmatrix}1&1&1\\2&2&2\\3&2&1\end{bmatrix}$, then

$$\begin{bmatrix}1&1&1\\2&2&2\\3&2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c}1&1&1&0\\2&2&2&0\\3&2&1&0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1&1&1&0\\0&0&0&0\\0&-1&-2&0\end{array}\right] \sim \begin{array}{c}\begin{array}{cccc}x&y&z&\end{array}\\ \left[\begin{array}{ccc|c}1&0&-1&0\\0&0&0&0\\0&1&2&0\end{array}\right]\end{array} \Rightarrow$$

$$\begin{cases}x-z=0\\ y+z=0\end{cases},$$ so let $z=t$ any #

$\Rightarrow$

$x = t$ and $y = -t$, then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \text{ a line}$$

c.) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$, then

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow x + y + z = 0$$ so

let $z = t$ any #,

let $y = s$ any # $\Rightarrow$

$x = -s - t$, then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

$$= s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \text{ a plane}$$

25.) a.) $3x - 5y = 0$ iff

$$\begin{bmatrix}3 & -5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = [0],$$ so all 2×2 matrices whose null space is the line $3x - 5y = 0$ are:

$$A = \begin{bmatrix}3c & -5c\\3d & -5d\end{bmatrix},$$ where $c \neq 0$ and d is any #

b.) $$\begin{bmatrix}1 & 4\\0 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \left[\begin{array}{cc|c}1 & 4 & 0\\0 & 5 & 0\end{array}\right] \sim \begin{array}{c}\begin{array}{cc}x & y\end{array}\\ \left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\end{array}\right]\end{array}$$

$\Rightarrow x = 0, y = 0$, so null space is $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ (a single point);

$$\begin{bmatrix}6 & 2\\3 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \left[\begin{array}{cc|c}6 & 2 & 0\\3 & 1 & 0\end{array}\right] \sim \begin{array}{c}\begin{array}{cc}x & y\end{array}\\ \left[\begin{array}{cc|c}0 & 0 & 0\\3 & 1 & 0\end{array}\right]\end{array}$$

$\Rightarrow 3x + y = 0$ so null space is all pts. on this line;

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{array}{c}\begin{array}{cc}x & y\end{array}\\ \left[\begin{array}{cc|c}0 & 0 & 0\\0 & 0 & 0\end{array}\right]\end{array} \Rightarrow$$

null space is all points in the xy-plane

27.) Assume A is an $n \times n$ invertible matrix. Show that the row vectors of A form a basis for $\mathbb{R}^n$:

Since A is invertible it is row equivalent to the $n \times n$ identity matrix I, i.e., there exist k elementary matrices $E_1, E_2, \dots, E_k$ so that

$$E_k \cdots E_2 E_1 A = I \quad .$$

By Theorem 4.74 the row space of A is the same as the row space of I. But the row space of

$$I = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ is $\mathbb{R}^n$, so the row space of A is $\mathbb{R}^n$.

Since A has n rows, the

rows of A are linearly independent and therefore form a basis for $\mathbb{R}^n$.

28.) Assume A and B are $n \times n$ matrices and A is invertible. Then the row space of AB is equal to the row space of B:

Since A is invertible it is row equivalent to the $n \times n$ identity matrix I, i.e. there exist k elementary matrices $E_1, E_2, \dots, E_k$ so that

$$E_k \cdots E_2 E_1 A = I \Rightarrow$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1},$$ where

$E_1^{-1}, E_2^{-1}, \dots,$ and E_k^{-1} are elementary matrices. Then

$$AB = E_1^{-1} E_2^{-1} \cdots E_k^{-1} B.$$

So by Theorem 4.74, the row space of AB is the same as the row space of B.

TRUE/FALSE

(a) T (b) F (c) F (d) F

(e) F (f) T (g) T (h) F

(i) T (j) F