Section 4.8

1.) a.)

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow n = 4$, $\text{rank}(A) = 1$, $\text{nullity}(A) = 3$

b.)

$$A = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix} \Rightarrow n = 5,$$

$\text{rank}(A) = 2$, $\text{nullity}(A) = 3$

2.) a.)

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & 1 & 3 & 0 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 1 & 3 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 4 \end{bmatrix}$$

$n=5$, rank$(A)=3$, nullity$(A)=2$

3.) a.) rank$(A) = 3$, nullity$(A) = 0$

b.) rank(A) + nullity$(A) = n \Rightarrow$

$3 + 0 = 3$

c.) zero parameters since

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

4.) a.) rank$(A)=2$, nullity$(A)=1$

b.) rank(A) + nullity$(A) = n \Rightarrow$

$2 + 1 = 3$

c.) one parameter since

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

5.) a.) rank$(A)=1$, nullity$(A)=2$

b.) rank(A) + nullity$(A) = n \Rightarrow$

$1 + 2 = 3$

c.) two parameters since

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -3/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

6.) a.) rank(A) = 3, nullity(A) = 1

b.) rank(A) + nullity(A) = n $\Rightarrow$

$$3 + 1 = 4$$

c.) one parameter since

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

7.) a.) 4; 0 b.) 3; 2 c.) 3; 0

8.) A is m (rows) x n (columns):

a.) largest possible rank is the minimum of m and n

b.) minimum possible nullity is $n - \min\{m, n\}$

9.		(a)	(b)	(c)	(d)	(e)	(f)	(g)
(i)	dimension of the row space of A	3	2	1	2	2	0	2
	dimension of the column space of A	3	2	1	2	2	0	2
	dimension of the null space of A	0	1	2	7	7	4	0
	dimension of the null space of A^T	0	1	2	3	3	4	4
(ii)	is the system $A\mathbf{x} = \mathbf{b}$ consistent?	Yes	No	Yes	Yes	No	Yes	Yes
(iii)	number of parameters in the general solution of $A\mathbf{x} = \mathbf{b}$	0	—	2	7	—	4	0

10.) $$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 7 & 17 & 2 \\ 0 & 7 & 17 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 7 & 17 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ so $\text{rank}(A) = 2$;

$$A^T = \begin{bmatrix} 1 & -3 & -2 \\ 2 & 1 & 3 \\ 4 & 5 & 9 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & 7 & 7 \\ 0 & 17 & 17 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

so $\text{rank}(A^T) = 2$

11) A is 3×4 and A^T is 4×3:

a.) $\text{nullity}(A) = 4 - \text{rank}(A)$

$= 4 - 2 = 2$;

$\text{nullity}(A^T) = 3 - \text{rank}(A^T)$

$= 3 - \text{rank}(A) = 3 - 2 = 1$;

then

$\text{nullity}(A) - \text{nullity}(A^T) = 1$

b.) A is $m \times n$ and A^T is $n \times m$:

$\text{nullity}(A) = n - \text{rank}(A)$;

$\text{nullity}(A^T) = m - \text{rank}(A^T)$

$= m - \text{rank}(A) \Rightarrow$

$\text{nullity}(A) - \text{nullity}(A^T) = n - m$

12.) $T(x_1, x_2) = (x_1 + 3x_2, x_1 - x_2, x_1)$;

then $T(1,0) = (1,1,1)$ and

$T(0,1) = (3,-1,0)$, so the

standard matrix for T is

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix},$$ so that

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ x_1 - x_2 \\ x_1 \end{bmatrix};$$

then $\begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & -4 \\ 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

a.) so $\text{rank}(A) = 2$

b.) $\text{nullity}(A) = 2 - \text{rank}(A)$

$= 2 - 2 = 0$

13.) $T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2, x_2 + x_3 + x_4, x_4 + x_5)$;

$T(1,0,0,0,0) = (1,0,0)$,

$T(0,1,0,0,0) = (1,1,0)$,

$T(0,0,1,0,0) = (0,1,0)$,

$T(0,0,0,1,0) = (0,1,1)$,

$T(0,0,0,0,1) = (0,0,1)$; so the standard matrix for T is

$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, so that

$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 + x_4 \\ x_4 + x_5 \end{bmatrix}$;

then $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, so that

a.) $\text{rank}(A) = 3$ and

b.) $\text{nullity}(A) = 5 - \text{rank}(A)$
$= 5 - 3 = 2$

14.) a.) $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 1-t & 1-t \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ if $\boxed{t=1}$ and has $\text{rank}\, A = 1$;

OR

$\sim \begin{bmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 0 & 2-2t \end{bmatrix}$ if $\boxed{t \neq 1}$ and has $\text{rank}\, A = 3$

b.) $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix} \sim \begin{bmatrix} 0 & 3-3t & t^2-1 \\ 0 & -3 & 3t-2 \\ -1 & -3 & t \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 1 \\ -1 & -3 & 1 \end{bmatrix}$ if $\boxed{t=1}$ and has $\text{rank}\, A = 2$;

OR
$$\sim \begin{bmatrix} 0 & 3(1-t) & (t-1)(t+1) \\ 0 & -3 & 3t-2 \\ -1 & -3 & t \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 3 & -t-1 \\ 0 & -3 & 3t-2 \\ -1 & -3 & t \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 2t-3 \\ 0 & -3 & 3t-2 \\ -1 & -3 & t \end{bmatrix} :$$

if $\boxed{t = 3/2}$, then rank $A = 2$;

if $\boxed{t \neq 3/2}$ and $t \neq 1$, then rank $A = 3$

15.) $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

a.) rank A cannot be 1 :
If rank $A = 1$, then 1st column generates the column space $\Rightarrow$ $r = 2$ and $s = 1$ so that the 2nd column is all 0's $\Rightarrow$ 3rd column is $\begin{bmatrix} 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}$, which

means the rank $A = 2$, a contradiction

b.) rank A can equal 2: If $r = 2$ and $s = 1$, then rank $A = 2$ (SEE part a.); for any other values of r and s the rank $A = 3$

16.) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$

a.) The column space of A is all linear combinations of the 2 nonzero column vectors, i.e.,

$$\alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix},$$ which is

plane through the origin.

b.) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \end{array} \Rightarrow$$

$x=0$, $y=0$, and $z=t$ any #, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$ which is a line through the origin (in particular, the z-axis)

17.) If A is 3×3 and its null space is a line through the origin $\Rightarrow$ nullity of $A = 1$

$\Rightarrow$ rank $A = 3 - 1 = 2$

$\Rightarrow$ column space is plane through the origin;

and column space and row space have the same dimension

$\Rightarrow$ row space is a plane through the origin

18.) a.) ... at most $\boxed{3}$, the minimum of 3 and 5

b.) ... at most $\boxed{5}$, the # of columns of A

c.) ... at most $\boxed{3}$, the minimum of 3 and 5

d.) ... at most $\boxed{3}$, the # of columns of A^T

19.) a.) ... at most $\boxed{3}$, the minimum of 3 and 5

b.) ... at most $\boxed{5}$, the # of columns of A

c.) ... at most $\boxed{3}$, the minimum of 3 and 5

d.) ... at most $\boxed{3}$, the # of columns of A

20.) $A = 7$ (rows) $\times$ 6 (columns), $A\vec{x} = \vec{0}$ has only $\vec{x} = \vec{0}$ as a solution

$\Rightarrow$ nullity of $A = 0$

$\Rightarrow$ rank $A = 6 - 0 = 6$

21.) A is 5 (rows) $\times$ 7 (columns)

and rank $A = 4$

a.) dimension of solution space of $A\vec{x} = \vec{0}$ is

nullity of $A = n - \text{rank } A$
$= 7 - 4 = 3$

b.) If $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$ is any vector in $\mathbb{R}^5$, then

solve $A\vec{x} = \vec{b} \Rightarrow$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{17} & b_1 \\ a_{21} & a_{22} & \cdots & a_{27} & b_2 \\ \vdots & & \ddots & & \vdots \\ a_{51} & a_{52} & \cdots & a_{57} & b_5 \end{array}\right] \sim \cdots$$

since rank $A = 4$

$$\sim \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & \text{—} & \text{—} & \text{—} & c_1 \\ 0 & 1 & 0 & 0 & \text{—} & \text{—} & \text{—} & c_2 \\ 0 & 0 & 1 & 0 & \text{—} & \text{—} & \text{—} & c_3 \\ 0 & 0 & 0 & 1 & \text{—} & \text{—} & \text{—} & c_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_5 \end{array}\right]$$

which may not be consistent

if $c_5 \neq 0$!

24.) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,

then $\text{rank } A = 1 = \text{rank } B$;

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A^2) = 0;$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(B^2) = 1.$$

28.)

$$\begin{array}{cc} x_1 & x_2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 1 & -2 & b_2 \\ 1 & 1 & b_3 \\ 1 & -4 & b_4 \\ 1 & 5 & b_5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 4 & b_3 - b_1 \\ 0 & -1 & b_4 - b_1 \\ 0 & 8 & b_5 - b_1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|l} 1 & 0 & 3(b_2 - b_1) + b_1 = 3b_2 - 2b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & -4(b_2 - b_1) + b_3 - b_1 = 3b_1 - 4b_2 + b_3 \\ 0 & 0 & (b_2 - b_1) + b_4 - b_1 = b_2 + b_4 - 2b_1 \\ 0 & 0 & -8(b_2 - b_1) + b_5 - b_1 = 7b_1 - 8b_2 + b_5 \end{array}\right]$$

$$\Rightarrow \begin{cases} 3b_1 - 4b_2 + b_3 = 0 \\ b_2 + b_4 - 2b_1 = 0 \\ 7b_1 - 8b_2 + b_5 = 0 \end{cases}$$

29.) The rank A is the # of nonzero rows after reducing A to row-echelon form. If $k \neq 0$ then start row reduction of matrix kA by 1st multiplying each row of kA by $1/k$. This reduces kA to A and will lead to the same row-echelon form. Thus rank A = rank (kA) if $k \neq 0$.

30.) Case 1: Assume A is $m \times n$, where $n > m$, say $n = m + k$, and n is the # of columns of A. But rank $A \leq \min(m, n) = m$ is the # of linearly independent columns of A. Since the # of columns of A is $m+k$, <u>at least k of the column vectors are linearly dependent</u> on the other vectors.

<u>Case 2</u>: assume A is $m \times n$, where $n < m$, say $m = n + k$, and m is the # of rows of A. But $\operatorname{rank} A \leq \min(m, n) = n$ is the # of linearly independent rows of A. Since the # of rows of A is $n+k$, <u>at least k of the row vectors are linearly dependent</u> on the other vectors.

33.) Assume $\vec{v} \in \mathbb{R}^n$. Let $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ be a basis for subspace W of $\mathbb{R}^n$. If $\vec{v} \cdot b_i = 0$ for all $i = 1, 2, \ldots, k$, then $\vec{v} \cdot \vec{w} = 0$ for all $\vec{w} \in W$: Since $\vec{w} \in W \Rightarrow$

$\vec{w} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_k b_k$

since $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ is a basis for W. Then

$$\vec{v} \cdot \vec{w} = \vec{v} \cdot (c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_k \vec{b}_k)$$

$$= \vec{v} \cdot (c_1 \vec{b}_1) + \vec{v} \cdot (c_2 \vec{b}_2) + \cdots + \vec{v} \cdot (c_k \vec{b}_k)$$

$= c_1(\vec{v}\cdot\vec{b_1}) + c_2(\vec{v}\cdot\vec{b_2}) + \cdots + c_k(\vec{v}\cdot\vec{b_k})$

$= c_1(0) + c_2(0) + \cdots + c_k(0)$

$= 0$, i.e., $\vec{v} \perp \vec{w}$.

TRUE/FALSE

(a) F (b) T (c) F

(d) F (e) T (f) F

(g) F (h) F (i) T

(j) F