Section 6.3

1.) a.) \( (0, 1) \cdot (2, 0) = 0 \times 2 + 1 \times 0 = 0 \),
   \( \| (2, 0) \| = 2 \), so ORTHOGONAL

b.) \( \left( \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \cdot \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \left( \frac{-1}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{2}} \right) + \left( \frac{1}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{2}} \right) = 0 \),
   \( \| (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \| = \sqrt{\left( \frac{-1}{\sqrt{2}} \right)^2 + \left( \frac{1}{\sqrt{2}} \right)^2} = \sqrt{1} = 1 \),
   \( \| (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \| = \sqrt{\left( \frac{1}{\sqrt{2}} \right)^2 + \left( \frac{1}{\sqrt{2}} \right)^2} = \sqrt{1} = 1 \),
   so ORTHONORMAL

c.) \( (0, 0) \cdot (0, 1) = 0 \times 0 + 0 \times 1 = 0 \),
   \( \| (0, 0) \| = 0 \), so ORTHOGONAL

2.) a.) \( \left( \frac{1}{\sqrt{2}}, 0 \right) \cdot \left( \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 \times \frac{-1}{\sqrt{2}} = 0 \),
   \( \left( \frac{1}{\sqrt{2}}, 0 \right) \cdot \left( \frac{-1}{\sqrt{2}}, 0 \right) = \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} + 0 \times 0 = 0 \),
   \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \cdot \left( \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} = \frac{-1}{2} \neq 0 \), so NOT ORTHOGONAL

c.) \( (1, 0, 0) \cdot (0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = 0 + 0 + 0 = 0 \),
   \( (1, 0, 0) \cdot (0, 0, 1) = 0 + 0 + 0 = 0 \)
\[
(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \cdot (0, 0, 1) = 0 + 0 + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \neq 0,
\]
so NOT ORTHOGONAL

3.) a.) \[\langle p_1(x), p_2(x) \rangle = \left( \frac{2}{3} \right) \left( \frac{2}{3} \right) + \left( -\frac{2}{3} \right) \left( \frac{2}{3} \right) + \left( \frac{1}{3} \right) \left( \frac{-2}{3} \right) = 0,\]
\[\langle p_1(x), p_3(x) \rangle = \left( \frac{2}{3} \right) \left( \frac{1}{3} \right) + \left( -\frac{2}{3} \right) \left( \frac{2}{3} \right) + \left( \frac{1}{3} \right) \left( \frac{2}{3} \right) = 0,\]
\[\langle p_2(x), p_3(x) \rangle = \left( \frac{2}{3} \right) \left( \frac{1}{3} \right) + \left( \frac{2}{3} \right) \left( \frac{2}{3} \right) + \left( -\frac{2}{3} \right) \left( \frac{2}{3} \right) = 0.\]

\[|| p_1(x) || = \sqrt{\left( \frac{2}{3} \right)^2 + \left( -\frac{2}{3} \right)^2 + \left( \frac{1}{3} \right)^2} = \sqrt{1} = 1,\]
\[|| p_2(x) || = \sqrt{\left( \frac{2}{3} \right)^2 + \left( \frac{1}{3} \right)^2 + \left( -\frac{2}{3} \right)^2} = \sqrt{1} = 1,\]
\[|| p_3(x) || = \sqrt{\left( \frac{1}{3} \right)^2 + \left( \frac{2}{3} \right)^2 + \left( \frac{2}{3} \right)^2} = \sqrt{1} = 1,\]
so ORTHONORMAL.

b.) \[\langle p_2(x), p_3(x) \rangle = 0(0) + \left( \frac{1}{\sqrt{2}} \right)(0) + \left( \frac{1}{\sqrt{2}} \right)(1) = \frac{1}{\sqrt{2}} \neq 0, \text{ so NOT ORTHOGONAL.}\]

4.) b.) \[\left[ \begin{array}{c} 1 \\ 0 \\ 3 \\ 8 \\ 9 \\ 3 \\ 1 \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{array} \right] = (1)(0) + (0)(0) + (0)(0) + (0)(0) + (0)(0) + (0)(0) = 0,\]
\[\left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{array} \right] = (1)(0) + (0)(0) + (0)(1) + (0)(0) + (0)(0) + (0)(1) = 0.\]
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
1 & -1
\end{bmatrix}
= (1)(0) + (0)(0) + (0)(1) + (0)(-1) = 0
\]
\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix}
= (0)(0) + (1)(0) + (0)(1) + (0)(0) = 0
\]
\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
1 & -1
\end{bmatrix}
= (0)(0) + (1)(0) + (0)(1) + (0)(-1) = 0
\]

\[
\left\| \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \right\| = \sqrt{(0)^2 + (0)^2 + (1)^2 + (0)^2} = \sqrt{1} = 1
\]
so ORTHONORMAL

7. \( \overrightarrow{v_1} \cdot \overrightarrow{v_2} = \left( \frac{3}{5} \right)(\frac{3}{5}) + \left( \frac{4}{5} \right)(\frac{3}{5}) + (0)(0) = 0 \)
\( \overrightarrow{v_1} \cdot \overrightarrow{v_3} = (-\frac{3}{5})(0) + (\frac{4}{5})(0) + (0)(1) = 0 \)
\( \overrightarrow{v_2} \cdot \overrightarrow{v_3} = (\frac{4}{5})(0) + (\frac{3}{5})(0) + (0)(1) = 0 \)

\[
\left\| \overrightarrow{v_1} \right\| = \sqrt{(-\frac{3}{5})^2 + (\frac{4}{5})^2 + (0)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1
\]
\[
\left\| \overrightarrow{v_2} \right\| = \sqrt{(\frac{4}{5})^2 + (\frac{3}{5})^2 + (0)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1
\]
\[
\left\| \overrightarrow{v_3} \right\| = \sqrt{(0)^2 + (0)^2 + (0)^2} = \sqrt{1} = 1
\]
so ORTHONORMAL
\[
\vec{u} = (1,-2,2) = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 \Rightarrow \\
\alpha_1 = (1,-2,2) \cdot (-\frac{3}{5}, \frac{4}{5}, 0) = -\frac{3}{5} + \frac{8}{5} = \frac{5}{5}
\]
\[
\alpha_2 = (1,-2,2) \cdot (4\frac{3}{5}, \frac{3}{5}, 0) = \frac{4}{5} - \frac{6}{5} = -\frac{2}{5}
\]
\[
\alpha_3 = (1,-2,2) \cdot (0,0,1) = 2
\]
\[
(1,-2,2) = (-\frac{11}{5}) (-\frac{3}{5}, \frac{4}{5}, 0) + (\frac{2}{5}) (\frac{4}{5}, \frac{3}{5}, 0) + (2) (0,0,1)
\]

9. \[
\vec{v}_1 \cdot \vec{v}_2 = (2)(2) + (-2)(0) + (1)(-2) = 0
\]
\[
\vec{v}_1 \cdot \vec{v}_3 = (2)(1) + (2)(2) + (1)(0) = 0
\]
\[
\vec{v}_2 \cdot \vec{v}_3 = (2)(1) + (1)(2) + (-2)(2) = 0
\]
\[
||\vec{v}_1||^2 = (2)^2 + (-2)^2 + (1)^2 = 9
\]
\[
||\vec{v}_2||^2 = (2)^2 + (1)^2 + (-2)^2 = 9
\]
\[
||\vec{v}_3||^2 = (1)^2 + (2)^2 + (2)^2 = 9
\]
so \(\vec{v}_1, \vec{v}_2, \vec{v}_3\) are orthogonal
\[
\vec{u} = (-1,0,2) = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 \Rightarrow \\
\alpha_1 = \frac{(-1,0,2) \cdot (2,-2,1)}{||\vec{v}_1||^2} = \frac{-2 + 2}{9} = 0
\]
\[ \alpha_2 = \frac{\langle -1,0,2 \rangle \cdot (2,1,-2) \rangle}{\| \vec{v}_2 \|^2} = \frac{-2 + 4}{9} = \frac{-2}{3} \]

\[ \alpha_3 = \frac{\langle -1,0,2 \rangle \cdot (1,2,2) \rangle}{\| \vec{v}_3 \|^2} = \frac{1 + 4}{9} = \frac{1}{3} \Rightarrow \]

\[ \langle -1,0,2 \rangle = \left( \frac{2}{3} \right) (2,1,-2) + \left( \frac{1}{3} \right) (1,2,2) \]

15. \( \vec{u} = (-1,6) \), \( \vec{v} = (3/5, 4/5) \)

a.) \( \text{proj}_\vec{v} \vec{u} = \left( \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v} \)

\[ = \left( \frac{-3/5 + 24/5}{3/25 + 16/25} \right) \vec{v} = \frac{21}{5} \vec{v} = \frac{21}{5} \left( \frac{3}{5}, \frac{4}{5} \right) \]

\[ = (\frac{63}{25}, \frac{84}{25}) \]

b.) \( \vec{u} - \text{proj}_\vec{v} \vec{u} = (-1,6) - (\frac{63}{25}, \frac{84}{25}) \)

\[ = \left( \frac{25}{25} - \frac{63}{25}, \frac{150}{25} - \frac{84}{25} \right) = \left( \frac{-38}{25}, \frac{66}{25} \right) \]

**CHECK:** \( \vec{v} \cdot (\vec{u} - \text{proj}_\vec{v} \vec{u}) \)

\[ = \left( \frac{3}{5}, \frac{4}{5} \right) \cdot \left( \frac{-38}{25}, \frac{66}{25} \right) = \frac{-264}{125} + \frac{264}{125} = 0 \]
18. \( \overrightarrow{u} = (3, -1), \overrightarrow{v} = (3, 4) \)

a.) \( \text{proj}_v \overrightarrow{u} = \left( \frac{\overrightarrow{u} \cdot \overrightarrow{v}}{\overrightarrow{v} \cdot \overrightarrow{v}} \right) \overrightarrow{v} \)

\[ = \frac{9 - 4}{9 + 16} \overrightarrow{v} = \frac{1}{5} (3, 4) = \left( \frac{3}{5}, \frac{4}{5} \right) \]

b.) \( \overrightarrow{u} - \text{proj}_v \overrightarrow{u} = (3, -1) - \left( \frac{3}{5}, \frac{4}{5} \right) \)

\[ = \left( \frac{15}{5} - \frac{3}{5}, \frac{5}{5} - \frac{4}{5} \right) = \left( \frac{12}{5}, \frac{-9}{5} \right) \]

CHECK: \( \overrightarrow{v} \cdot (\overrightarrow{u} - \text{proj}_v \overrightarrow{u}) \)

\[ = (3, 4) \cdot \left( \frac{12}{5}, \frac{-9}{5} \right) = \frac{36}{5} + \frac{-36}{5} = 0 \]

21. \( \overrightarrow{u} = (1, 0, 3), \overrightarrow{v}_1 = (1, -2, 1), \overrightarrow{v}_2 = (2, 1, 0) \)

\( W = \text{span} \{ \overrightarrow{v}_1, \overrightarrow{v}_2 \} \)

a.) \( \text{proj}_W \overrightarrow{u} = \left( \frac{\overrightarrow{u} \cdot \overrightarrow{v}_1}{\overrightarrow{v}_1 \cdot \overrightarrow{v}_1} \right) \overrightarrow{v}_1 + \left( \frac{\overrightarrow{u} \cdot \overrightarrow{v}_2}{\overrightarrow{v}_2 \cdot \overrightarrow{v}_2} \right) \overrightarrow{v}_2 \)

\[ = \frac{1 + 3}{1 + 4 + 1} \overrightarrow{v}_1 + \frac{2}{4 + 1} \overrightarrow{v}_2 = \frac{2}{3} (1, -2, 1) + \frac{2}{5} (2, 1, 0) \]

\[ = \left( \frac{2}{3}, \frac{-4}{3}, \frac{2}{3} \right) + \left( \frac{4}{5}, \frac{2}{5}, 0 \right) \]

\[ = \left( \frac{10}{15} + \frac{12}{15} - \frac{20}{15} + \frac{6}{15}, \frac{2}{3} + 0 \right) = \left( \frac{22}{15}, \frac{-14}{15}, \frac{2}{3} \right) \]
b. \( \vec{u} - \text{proj}_W \vec{u} = (1, 0, 3) - \left( \frac{22}{15} \cdot \frac{14}{15}, \frac{9}{3}, -\frac{2}{3} \right) = \left( \frac{15}{15} - \frac{22}{15}, \frac{9}{3}, -\frac{2}{3} \right) = \left( -\frac{7}{15}, \frac{14}{15}, \frac{7}{15} \right) \)

**CHECK:** \( \vec{v}_1 \cdot (\vec{u} - \text{proj}_W \vec{u}) \)

\[ (1, -2, 1) \cdot \left( -\frac{7}{15}, \frac{14}{15}, \frac{7}{15} \right) = -\frac{7}{15} - \frac{28}{15} + \frac{35}{15} = 0 \]

\( \vec{v}_2 \cdot (\vec{u} - \text{proj}_W \vec{u}) \)

\[ (2, 1, 0) \cdot \left( -\frac{7}{15}, \frac{14}{15}, \frac{7}{15} \right) = -\frac{14}{15} + \frac{14}{5} = 0 \]

22. \( \vec{u} = (1, 0, 2), \vec{v}_1 = (3, 1, 2), \vec{v}_2 = (-1, 1, 1) \)

\( W = \text{span} \{ \vec{v}_1, \vec{v}_2 \} \)

Since \( \vec{v}_1 \cdot \vec{v}_2 = 0 \)

\( \text{a.} \quad \text{proj}_W \vec{u} = \left( \frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 + \left( \frac{\vec{u} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \right) \vec{v}_2 \)

\[ \frac{3+4}{9+1+4} \vec{v}_1 + \frac{-1+2}{1+1+1} \vec{v}_2 \]

\[ = \frac{1}{2} (3, 1, 2) + \frac{1}{3} (-1, 1, 1) = \left( \frac{3}{2}, \frac{1}{2}, 1 \right) + \left( -\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \]

\[ = \left( \frac{9}{6} - \frac{2}{6} \cdot \frac{3}{6} + \frac{2}{6} \cdot \frac{3}{3} + \frac{1}{3} \right) = \left( \frac{7}{6}, \frac{5}{6}, \frac{4}{3} \right) \]
6. \( \mathbf{u} - \text{proj}_W \mathbf{u} = (1,0,2) - \left( \frac{1}{6} \mathbf{i} + \frac{5}{6} \mathbf{j} + \frac{4}{3} \mathbf{k} \right) \\
\quad = \left( \frac{6}{6} - \frac{1}{6}, \frac{5}{6} - \frac{4}{6}, \frac{4}{3} - \frac{4}{3} \right) = \left( \frac{5}{6}, \frac{1}{6}, 0 \right) \\
\text{CHECK: } \mathbf{v}_1 \cdot (\mathbf{u} - \text{proj}_W \mathbf{u}) \\
\quad = (3,1,2) \cdot \left( \frac{5}{6}, \frac{1}{6}, 0 \right) = \frac{15}{6} + \frac{1}{6} + 0 = 3 \\
\mathbf{v}_2 \cdot (\mathbf{u} - \text{proj}_W \mathbf{u}) \\
\quad = (-1,1,1) \cdot \left( \frac{5}{6}, \frac{1}{6}, 0 \right) = \frac{-5}{6} + \frac{1}{6} + 0 = 0 \\
23. \mathbf{u} = (1,2,0,2), \mathbf{v}_1 = (1,1,1), \mathbf{v}_2 = (1,1,1,1)

\text{W} = \text{span} \{ \mathbf{v}_1, \mathbf{v}_2 \} \quad \text{since } \mathbf{v}_1 \cdot \mathbf{v}_2 = 0

\text{then}
\text{proj}_W \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \left( \frac{\mathbf{u} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2

= \frac{1+2-2}{1+1+1+1} \mathbf{v}_1 + \frac{1+2+2}{1+1+1+1} \mathbf{v}_2

= \frac{1}{4} \cdot (1,1,1) + \frac{5}{4} \cdot (1,1,1,1)

= \left( \frac{1}{4} \cdot 1, \frac{1}{4} \cdot 1, \frac{1}{4} \cdot 1, \frac{1}{4} \cdot 1, \frac{5}{4} \cdot 1, \frac{5}{4} \cdot 1, \frac{5}{4} \cdot 1, \frac{5}{4} \cdot 1 \right)

= \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right)
26. \( \overrightarrow{u} = (1, 2, 3, -1) \), \( \overrightarrow{v}_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \),
\( \overrightarrow{v}_2 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) \), \( \overrightarrow{v}_3 = (\frac{1}{2}, 1, -\frac{1}{2}, -\frac{1}{2}) \);
\( \overrightarrow{v}_1 \cdot \overrightarrow{v}_2 = 0 \), \( \overrightarrow{v}_1 \cdot \overrightarrow{v}_3 = 0 \), \( \overrightarrow{v}_2 \cdot \overrightarrow{v}_3 = 0 \),
\( \| \overrightarrow{v}_1 \| = \| \overrightarrow{v}_2 \| = \| \overrightarrow{v}_3 \| = \frac{\sqrt{4 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}}{2} = 1 \),
so \( \overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3 \) are ORTHONORMAL.
\( W = \text{span} \{ \overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3 \} \), then
\[
\text{proj}_W \overrightarrow{u} = (\overrightarrow{u} \cdot \overrightarrow{v}_1) \overrightarrow{v}_1 + (\overrightarrow{u} \cdot \overrightarrow{v}_2) \overrightarrow{v}_2 + (\overrightarrow{u} \cdot \overrightarrow{v}_3) \overrightarrow{v}_3
\]
\[
= (\frac{1}{2} + \frac{3}{2} - \frac{1}{2}) \overrightarrow{v}_1 + (\frac{1}{2} + \frac{3}{2} + \frac{1}{2}) \overrightarrow{v}_2 + (\frac{1}{2} - \frac{3}{2} + \frac{1}{2}) \overrightarrow{v}_3
\]
\[
= (1) \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) + (2) \left( \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right) + (0) \left( \frac{1}{2}, 1, -\frac{1}{2}, -\frac{1}{2} \right)
\]
\[
= \left( \frac{1}{2} + \frac{3}{2}, \frac{1}{2} - \frac{3}{2}, \frac{1}{2} - \frac{3}{2}, \frac{1}{2} \right) = \left( \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)
\]

27. \( \overrightarrow{u}_1 = (1, -3) \), \( \overrightarrow{u}_2 = (2, 2) \)
\( i.) \overrightarrow{v}_1 = \overrightarrow{u}_1 = (1, -3) \)
\( ii.) \overrightarrow{v}_2 = \overrightarrow{u}_2 - \left( \frac{\overrightarrow{u}_2 \cdot \overrightarrow{v}_1}{\overrightarrow{v}_1 \cdot \overrightarrow{v}_1} \right) \overrightarrow{v}_1 \)
\[
= \overrightarrow{u}_2 - \frac{2 - 6}{1 + 9} \overrightarrow{v}_1 = (2, 2) - \frac{2}{5} (1, -3)
\]
\[
= \left( \frac{10}{5}, \frac{10}{5}, \frac{10}{5}, -\frac{6}{5} \right) = \left( \frac{12}{5}, \frac{4}{5} \right) \Rightarrow 
\]
now make unit vectors:

\[
\frac{1}{\| \mathbf{v}_1 \|} \mathbf{v}_1 = \frac{1}{\sqrt{1+9}} (1, -3) = \left( \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} \right)
\]

\[
\frac{1}{\| \mathbf{v}_2 \|} \mathbf{v}_2 = \frac{1}{\sqrt{\frac{144}{25} + \frac{16}{25}}} (\frac{12}{5}, \frac{4}{5})
\]

\[
= \frac{1}{\sqrt{\frac{160}{25}}} \left( \frac{12}{5}, \frac{4}{5} \right) = \frac{5}{4\sqrt{10}} \left( \frac{12}{5}, \frac{4}{5} \right)
\]

\[
= \left( \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right)
\]

28.) \( \mathbf{u}_1 = (1, 0), \mathbf{u}_2 = (3, -5) \)

i.) \( \mathbf{v}_1 = \mathbf{u}_1 = (1, 0) \)

ii.) \( \mathbf{v}_2 = \mathbf{u}_2 - \left( \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 \)

\[
= \mathbf{u}_2 - \frac{3+0}{1+0} \mathbf{v}_1 = (3, -5) - 3 (1, 0)
\]

\[= (0, -5) \Rightarrow \text{now make unit vectors:} \]

\[
\frac{1}{\| \mathbf{v}_1 \|} \mathbf{v}_1 = \left[ \begin{array}{c}
\frac{1}{\sqrt{10}} \\
\frac{-3}{\sqrt{10}}
\end{array} \right]
\]

\[
\frac{1}{\| \mathbf{v}_2 \|} \mathbf{v}_2 = \frac{1}{5} (0, -5) = (0, -1)
\]
29.) \( \vec{u}_1 = (1, 1, 1), \ \vec{u}_2 = (-1, 1, 0), \ \vec{u}_3 = (1, 2, 1) \)

i.) \( \vec{v}_1 = \vec{u}_1 = (1, 1, 1) \)

ii.) \( \vec{v}_2 = \vec{u}_2 - \left( \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 \)

\[ = \vec{u}_2 - \frac{-1+1}{1+1+1} \vec{v}_1 \]

\[ = (-1, 1, 0) - (0)(1, 1, 1) \]

\[ = (-1, 1, 0) \]

iii.) \( \vec{v}_3 = \vec{u}_3 - \left( \frac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \right) \vec{v}_2 - \left( \frac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 \)

\[ = \vec{u}_3 - \frac{-1+2}{1+1} \vec{v}_2 - \frac{1+2+1}{1+1+1} \vec{v}_1 \]

\[ = (1, 2, 1) - \frac{1}{2} (-1, 1, 0) - \frac{4}{3} (1, 1, 1) \]

\[ = \left( \frac{6+3-8}{6}, \frac{-3-2-3}{6}, \frac{6-8}{6} \right) \]

\[ = \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{3} \right) \Rightarrow \text{Now make unit vectors:} \]

\[ \frac{1}{||\vec{v}_1||} \vec{v}_1 = \frac{1}{\sqrt{1+1+1}} (1, 1, 1) = \frac{1}{\sqrt{3}} \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \]

\[ \frac{1}{||\vec{v}_2||} \vec{v}_2 = \frac{1}{\sqrt{1+1+1}} (-1, 1, 0) = \frac{1}{\sqrt{2}} \left( \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \]
\[
\frac{1}{\sqrt{3}} \vec{v}_3 = \frac{1}{\sqrt{\frac{1}{36} + \frac{1}{36} + \frac{4}{36}}} \left( \frac{1}{6} \right) \left( \begin{array}{c} -1 \\ 6 \\ 3 \end{array} \right)
\]

\[
= \sqrt{\frac{36}{6}} \left( \frac{1}{6} \right) \left( \begin{array}{c} -1 \\ 6 \\ 6 \end{array} \right) = \left( \frac{1}{\sqrt{6}} \right) \left( \begin{array}{c} -2 \\ 6 \\ 6 \end{array} \right)
\]

32. \( \vec{u}_1 = (0, 1, 2), \vec{u}_2 = (-1, 0, 1), \vec{u}_3 = (-1, 1, 3) \):

First find basis for subspace:

\[
\left[ \begin{array}{ccc} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -1 & 1 & 3 \end{array} \right]
\]

So basis is \( \{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \} \):

c.) \( \vec{v}_1 = \vec{u}_1 = (0, 1, 2) \)

d.) \( \vec{v}_2 = \vec{u}_2 - \left( \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 \)

\[
= \vec{u}_2 - \frac{2}{1+4} \vec{v}_1 = (1, 0, 1) - \frac{2}{5} (0, 1, 2)
\]

\[
= (-1, -\frac{2}{5}, \frac{1}{5}) \Rightarrow \text{now make unit vectors:}
\]

\[
\frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{\sqrt{1+4}} (0, 1, 2) = \left( 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)
\]
\[ \frac{1}{||\vec{v}_2||} \overrightarrow{v}_2 = \frac{1}{\sqrt{1 + \frac{4}{25} + \frac{1}{25}}} \left( \begin{array}{c} -1 \\
\frac{-2}{5} \\
\frac{1}{5} \end{array} \right) \]

\[ = \frac{1}{\sqrt{\frac{30}{25}}} \left( \begin{array}{c} -1 \\
\frac{-2}{5} \\
\frac{1}{5} \end{array} \right) = \frac{1}{\sqrt{\frac{30}{25}}} \left( \begin{array}{c} -5 \\
-2 \\
1 \end{array} \right) \]

\[ = \left( \frac{-5}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}} \right) \]

33. \( \vec{u} = (1, 2, 0, -2), \vec{v}_1 = (1, 1, 1, 1), \vec{v}_2 = (1, 1, -1, -1) \);
\( \{ \vec{v}_1, \vec{v}_2 \} \) is an orthogonal basis

for \( W = \text{span} \{ \vec{v}_1, \vec{v}_2 \} \); if

\[ \vec{u} = \vec{w}_1 + \vec{w}_2 \], where \( \vec{w}_1 \in W \) and
\( \vec{w}_2 \in W^\perp \): then

\[ \vec{w}_1 = \text{proj}_W \vec{u} \]

\[ = \left( \frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 + \left( \frac{\vec{u} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \right) \vec{v}_2 \]

\[ = \frac{1+2-2}{1+1+1+1} \vec{v}_1 + \frac{1+2+2}{1+1+1+1} \vec{v}_2 \]

\[ = \frac{1}{4} \left( \vec{v}_1 + \vec{v}_2 \right) + \frac{5}{4} \left( \vec{v}_1 - \vec{v}_2 \right) = \left( \frac{3}{2}, \frac{3}{2}, 1, -1 \right) \]
\[ \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 \]
\[ = (1, 2, 0, -2) - (3, 3, -1, -1) \]
\[ = (-3/2, 1, 1, -1) \in W \text{ since} \]
\[ \mathbf{w}_2 \cdot \mathbf{v}_1 = 0 \text{ and } \mathbf{w}_2 \cdot \mathbf{v}_2 = 0. \]

(\star) 43.) \( \mathbf{u}_1 = 1 \), \( \mathbf{u}_2 = x \), \( \mathbf{u}_3 = x^2 \), define \( p(x) \cdot q(x) = \int_0^1 p(x) q(x) \, dx \):

i.) \( \mathbf{v}_1 = \mathbf{u}_1 = 1 \)

ii.) \( \mathbf{v}_2 = \mathbf{u}_2 - \left( \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 \)

\[ = \mathbf{u}_2 - \frac{\int_0^1 x \, dx}{\int_0^1 1 \, dx} \mathbf{v}_1 \]
\[ = \mathbf{u}_2 - \frac{\frac{1}{2}x^2 \bigg|_0^1}{x \bigg|_0^1} \mathbf{v}_1 \]
\[ = \mathbf{u}_2 - \frac{1/2}{1} \mathbf{v}_1 = x - \frac{1}{2} \]

iii.) \( \mathbf{v}_3 = \mathbf{u}_3 - \left( \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 - \left( \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 \)
\[
\begin{align*}
\overrightarrow{u}_3 &= \overrightarrow{u}_3 - \frac{\int_0^1 x^2 (1-x) \, dx}{\int_0^1 (x-\frac{1}{2})^2 \, dx} \overrightarrow{v}_2 - \frac{\int_0^1 x^2 \, dx}{\int_0^1 1 \, dx} \overrightarrow{v}_1 \\
&= \overrightarrow{u}_3 - \frac{\int_0^1 (x^2 - x^3) \, dx}{\int_0^1 (x^2 - x + \frac{1}{4}) \, dx} \overrightarrow{v}_2 - \frac{\frac{1}{3} \int_0^1 x^3 \, dx}{\int_0^1 x \, dx} \overrightarrow{v}_1 \\
&= \overrightarrow{u}_3 - \frac{\left(\frac{4}{3} \frac{3}{4} x^4 \right)_0^1}{\left(\frac{1}{3} \frac{3}{2} x^2 + \frac{1}{4} x \right)_0^1} \overrightarrow{v}_2 - \frac{1}{3} \overrightarrow{v}_1 \\
&= \overrightarrow{u}_3 - \frac{\frac{1}{12}}{\frac{1}{12}} \overrightarrow{v}_2 - \frac{1}{3} \overrightarrow{v}_1 \\
&= x^2 - \left(x - \frac{1}{2}\right) - \frac{1}{3} (1) \\
&= x^2 - x + \frac{1}{6} \Rightarrow \text{now make unit vectors:}
\end{align*}
\]

\[
\frac{1}{\|\overrightarrow{v}_1\|} \overrightarrow{v}_1 = \frac{1}{\sqrt{\int_0^1 1 \, dx}} (1) = \boxed{1} \quad \overrightarrow{j}
\]

\[
\frac{1}{\|\overrightarrow{v}_2\|} \overrightarrow{v}_2 = \frac{1}{\sqrt{\int_0^1 (x-\frac{1}{2})^2 \, dx}} \overrightarrow{v}_2
\]

\[
= \frac{1}{\sqrt{\frac{1}{4}}} \overrightarrow{v}_2 = \boxed{2\sqrt{3} (x-\frac{1}{2})} \quad \overrightarrow{j}
\]
\[ \frac{1}{\| \mathbf{v}_3 \|} \mathbf{v}_3 = \frac{1}{\sqrt{\int_0^1 (x^2-x+\frac{1}{6})^2 \, dx}} \mathbf{v}_3 \]

\[ = \frac{1}{\sqrt{\frac{1}{3} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}}} \mathbf{v}_3 \]

\[ = \frac{1}{\sqrt{\frac{36}{180} - \frac{90}{180} + \frac{80}{180} - \frac{30}{180} + \frac{5}{180}}} \mathbf{v}_3 \]

\[ = \frac{1}{\sqrt{\frac{1}{180}}} \mathbf{v}_3 = 6\sqrt{5} (x^2-x+\frac{1}{6}) \]

**TRUE/ FALSE**

(c) T (e) F

55. Assume \( W \) is subspace of vector space \( V \) (finite dimension). Define
$T : V \to W$ by $T(\vec{v}) = \text{proj}_W \vec{v}$.

a.) Show $T$ is a linear transformation, i.e., show
   c.) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
   for all $\vec{u}, \vec{v} \in V$

   ii.) $T(\alpha \vec{v}) = \alpha T(\vec{v})$ for all $\vec{v} \in V$, constants $\alpha$

   i.) Let $\vec{e}_1, \vec{e}_2, ..., \vec{e}_k$ be an orthonormal basis for $W$. Then for $\vec{u}, \vec{v} \in V$ we have

   $T(\vec{u} + \vec{v}) = \text{proj}_W (\vec{u} + \vec{v})$

   $= \left( \frac{(\vec{u} + \vec{v}) \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 + \cdots + \left( \frac{(\vec{u} + \vec{v}) \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k} \right) \vec{v}_k$

   $= \left( \frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 + \cdots$

   $+ \left( \frac{\vec{u} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k} \right) \vec{v}_k + \left( \frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 + \cdots$

   $+ \left( \frac{\vec{v} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k} \right) \vec{v}_k$
\[
\begin{align*}
\mathbf{u} & = \left( \frac{\mathbf{u} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \ldots + \left( \frac{\mathbf{u} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \right) \mathbf{v}_k \\
+ \left( \frac{\mathbf{v} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 & + \ldots + \left( \frac{\mathbf{v} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \right) \mathbf{v}_k \\
& = \text{proj}_W \mathbf{u} + \text{proj}_W \mathbf{v}
\end{align*}
\]

(ii.) For \( \mathbf{v} \in V \) and constant \( \alpha \) we have
\[
\mathbf{1}(\alpha \mathbf{v}) = \text{proj}_W (\alpha \mathbf{v})
\]
\[
\begin{align*}
&= \left( \frac{\alpha \mathbf{v} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \ldots + \left( \frac{\alpha \mathbf{v} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \right) \mathbf{v}_k \\
&= \alpha \left( \frac{\mathbf{v} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \ldots + \alpha \left( \frac{\mathbf{v} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \right) \mathbf{v}_k \\
&= \alpha \left[ \frac{\mathbf{v} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \ldots + \frac{\mathbf{v} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \mathbf{v}_k \right] \\
&= \alpha \text{proj}_W \mathbf{v}
\end{align*}
\]
b.) Kernel of $T = \{ \vec{v} \in V | T(\vec{v}) = \vec{0} \}$; 
Show Kernel of $T = W^\perp$, where 
$W^\perp = \{ \vec{v} \in V | \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W \}$.

(2) Show $W^\perp \subseteq \text{Kernel of } T$:
If $\vec{v} \in W^\perp$ \implies $\vec{v} \cdot \vec{w} = 0$ for all $\vec{w} \in W$
\implies $\vec{v} \cdot \vec{v}_1 = 0, \vec{v} \cdot \vec{v}_2 = 0, \ldots$ and $\vec{v} \cdot \vec{v}_k = 0$ since $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in W$
\implies $T(\vec{v}) = \text{proj}_W \vec{v}$

$= \left( \frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 + \ldots + \left( \frac{\vec{v} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k} \right) \vec{v}_k$

$= (0) \vec{v}_1 + \ldots + (0) \vec{v}_k = \vec{0} \implies$ 
$\vec{v} \in \text{Kernel of } T$

(3) Show Kernel of $T \subseteq W^\perp$.
First extend the basis $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \}$ for $W$ to an
orthonormal basis
\[ \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{u}_{k+1}, \vec{u}_{k+2}, \ldots, \vec{u}_n \} \]
where \( \vec{u}_{k+1}, \vec{u}_{k+2}, \ldots, \vec{u}_n \in W^\perp. \)

Now let \( \vec{v} \in \text{Kernel of } T \Rightarrow \)
\[ \vec{v} = \alpha_1 \vec{v}_1 + \ldots + \alpha_k \vec{v}_k + \alpha_{k+1} \vec{u}_{k+1} + \ldots + \alpha_n \vec{u}_n \quad \text{and} \]
\[ T(\vec{v}) = \vec{0} \Rightarrow \]
\[ T(\vec{v}) = \text{proj}_W \vec{v} \]
\[ = \left( \frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 + \ldots + \left( \frac{\vec{v} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k} \right) \vec{v}_k \]
\[ = \frac{\alpha_1 \vec{v}_1 + \ldots + \alpha_k \vec{v}_k + \alpha_{k+1} \vec{u}_{k+1} + \ldots + \alpha_n \vec{u}_n}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \ldots + \frac{\alpha_1 \vec{v}_1 + \ldots + \alpha_k \vec{v}_k + \alpha_{k+1} \vec{u}_{k+1} + \ldots + \alpha_n \vec{u}_n}{\vec{v}_k \cdot \vec{v}_k} \vec{v}_k \]
\[ = \alpha_1 \frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \ldots + 0 + 0 + \ldots + 0 \vec{v}_1 + \ldots + \alpha_k \frac{\vec{v} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k} \vec{v}_k + \ldots + 0 + 0 + \ldots + 0 \vec{v}_k \]
\[ \vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_k \vec{v}_k = 0 \]

\[ \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0 \] (since \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) are linearly independent)

\[ \Rightarrow \vec{v} = \alpha_{k+1} \vec{u}_{k+1} + \cdots + \alpha_r \vec{u}_r \in W^\perp. \]

This completes the proof.

---

Range of \( T = \{ \vec{w} \in W \mid T(\vec{v}) = \vec{w} \} \) for some \( \vec{v} \in V \).

Show Range of \( T = W \): (\( \subseteq \) Show Range of \( T \subseteq W \): Let \( \vec{w} \in \text{Range of } T \), then \( \vec{w} \in W \).

(\( \supseteq \) Show \( W \subseteq \text{Range of } T \): Let \( \vec{w} \in W \), then

\[ T(\vec{w}) = \text{proj}_W \vec{w} = \vec{w} \Rightarrow \vec{w} \in \text{Range of } T. \]

This completes the proof.