Section 6.3

1.) a.) $\overrightarrow{(0,1)} \cdot \overrightarrow{(2,0)} = (0)(2) + (1)(0) = 0,$
$\|\overrightarrow{(2,0)}\| = 2,$ so ORTHOGONAL

b.) $\overrightarrow{\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} \cdot \overrightarrow{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} = \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 0,$

$\left\|\overrightarrow{\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}\right\| = \sqrt{\left(\frac{-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1} = 1,$

$\left\|\overrightarrow{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}\right\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1} = 1,$

so ORTHONORMAL

c.) $\overrightarrow{(0,0)} \cdot \overrightarrow{(0,1)} = (0)(0) + (0)(1) = 0,$
$\|\overrightarrow{(0,0)}\| = 0,$ so ORTHOGONAL

2.) a.) $\overrightarrow{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)} \cdot \overrightarrow{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)} = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0,$

$\overrightarrow{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)} \cdot \overrightarrow{\left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)} = \frac{-1}{2} + \frac{1}{2} = 0,$

$\overrightarrow{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)} \cdot \overrightarrow{\left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)} = \frac{-1}{\sqrt{6}} + 0 + \frac{-1}{\sqrt{6}}$

$= \frac{-2}{\sqrt{6}} \neq 0,$ so NOT ORTHOGONAL

c.) $\overrightarrow{(1,0,0)} \cdot \overrightarrow{\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} = 0 + 0 + 0 = 0,$

$\overrightarrow{(1,0,0)} \cdot \overrightarrow{(0,0,1)} = 0 + 0 + 0 = 0,$

$\overrightarrow{\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} \cdot \overrightarrow{(0,0,1)} = 0 + 0 + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \neq 0,$

so NOT ORTHOGONAL

3.) a.) $\langle p_1(x), p_2(x) \rangle = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{-2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right) = 0,$

$\langle p_1(x), p_3(x) \rangle = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{-2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 0,$

$\langle p_2(x), p_3(x) \rangle = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{-2}{3}\right)\left(\frac{2}{3}\right) = 0;$

$\|\overrightarrow{p_1(x)}\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{1}{3}\right)} = \sqrt{1} = 1,$

$\|p_2(x)\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{-2}{3}\right)^2} = \sqrt{1} = 1,$

$\|p_3(x)\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{1} = 1,$

so ORTHONORMAL

b.) $\langle p_2(x), p_3(x) \rangle = (0)(0) + \left(\frac{1}{\sqrt{2}}\right)(0) + \left(\frac{1}{\sqrt{2}}\right)(1)$

$= \frac{1}{\sqrt{2}} \neq 0,$ so NOT ORTHOGONAL

4.) b.) $\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\rangle = (1)(0) + (0)(1)$

$+ (0)(0) + (0)(0) = 0,$

$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\rangle = (1)(0) + (0)(0)$

$+ (0)(1) + (0)(1) = 0,$

$$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\rangle = (1)(0) + (0)(0)$$
$$+ (0)(1) + (0)(-1) = 0,$$
$$\left\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\rangle = (0)(0) + (1)(0)$$
$$+ (0)(1) + (0)(1) = 0,$$
$$\left\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\rangle = (0)(0) + (1)(0)$$
$$+ (0)(1) + (0)(-1) = 0,$$
$$\left\| \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\| = \sqrt{(0)^2 + (0)^2 + (1)^2 + (1)^2} = \sqrt{2} \neq 1,$$
~~so~~ ORTHOGONAL

7.) $\vec{v}_1 \cdot \vec{v}_2 = \left(-\frac{3}{5}\right)\left(\frac{4}{5}\right) + \left(\frac{4}{5}\right)\left(\frac{3}{5}\right) + (0)(0) = 0,$

$\vec{v}_1 \cdot \vec{v}_3 = \left(-\frac{3}{5}\right)(0) + \left(\frac{4}{5}\right)(0) + (0)(1) = 0,$

$\vec{v}_2 \cdot \vec{v}_3 = \left(\frac{4}{5}\right)(0) + \left(\frac{3}{5}\right)(0) + (0)(1) = 0,$

$\|\vec{v}_1\| = \sqrt{\left(-\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + (0)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1,$

$\|\vec{v}_2\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + (0)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1,$

$\|\vec{v}_3\| = \sqrt{(0)^2 + (0)^2 + (1)^2} = \sqrt{1} = 1,$

so ORTHONORMAL ;

$$\vec{u} = \overrightarrow{(1,-2,2)} = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \alpha_3 \vec{v_3} \Rightarrow$$

$$\alpha_1 = \overrightarrow{(1,-2,2)} \cdot \overrightarrow{(-3/5, 4/5, 0)} = \frac{-3}{5} + \frac{-8}{5} = \frac{-11}{5},$$

$$\alpha_2 = \overrightarrow{(1,-2,2)} \cdot \overrightarrow{(4/5, 3/5, 0)} = \frac{4}{5} + \frac{-6}{5} = \frac{-2}{5},$$

$$\alpha_3 = \overrightarrow{(1,-2,2)} \cdot \overrightarrow{(0,0,1)} = 2 \Rightarrow$$

$$\overrightarrow{(1,-2,2)} = \left(\frac{-11}{5}\right) \overrightarrow{(-3/5, 4/5, 0)}$$

$$+ \left(\frac{-2}{5}\right) \overrightarrow{\left(\frac{4}{5}, 3/5, 0\right)} + (2)\overrightarrow{(0,0,1)}$$

9.) $\vec{v_1} \cdot \vec{v_2} = (2)(2) + (-2)(1) + (1)(-2) = 0,$

$\vec{v_1} \cdot \vec{v_3} = (2)(1) + (-2)(2) + (1)(2) = 0,$

$\vec{v_2} \cdot \vec{v_3} = (2)(1) + (1)(2) + (-2)(2) = 0,$

$\|\vec{v_1}\|^2 = (2)^2 + (-2)^2 + (1)^2 = 9,$

$\|\vec{v_2}\|^2 = (2)^2 + (1)^2 + (-2)^2 = 9,$

$\|\vec{v_3}\|^2 = (1)^2 + (2)^2 + (2)^2 = 9,$

so ORTHOGONAL ;

$$\vec{u} = \overrightarrow{(-1,0,2)} = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \alpha_3 \vec{v_3} \Rightarrow$$

$$\alpha_1 = \frac{\overrightarrow{(-1,0,2)} \cdot \overrightarrow{(2,-2,1)}}{\|\vec{v_1}\|^2} = \frac{-2+2}{9} = 0,$$

$$\alpha_2 = \frac{\overrightarrow{(-1,0,2)} \cdot \overrightarrow{(2,1,-2)}}{\|\vec{v}_2\|^2} = \frac{-2+-4}{9} = \frac{-2}{3},$$

$$\alpha_3 = \frac{\overrightarrow{(-1,0,2)} \cdot \overrightarrow{(1,2,2)}}{\|\vec{v}_3\|^2} = \frac{-1+4}{9} = \frac{1}{3} \Rightarrow$$

$$\overrightarrow{(-1,0,2)} = (0)\,\overrightarrow{(2,-2,1)} + \left(\frac{-2}{3}\right)\overrightarrow{(2,1,-2)}$$
$$+ \left(\frac{1}{3}\right)\overrightarrow{(1,2,2)}$$

15.) $\vec{u} = \overrightarrow{(-1,6)}$, $\vec{v} = \overrightarrow{(3/5, 4/5)}$

a.) $\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$

$$= \left(\frac{-3/5 + 24/5}{9/25 + 16/25}\right)\vec{v} = \frac{\frac{21}{5}}{25/25}\,\vec{v} = \frac{21}{5}\,\overrightarrow{(3/5, 4/5)}$$

$$= \overrightarrow{\left(\frac{63}{25}, \frac{84}{25}\right)}$$

b.) $\vec{u} - \text{proj}_{\vec{v}}\vec{u} = \overrightarrow{(-1,6)} - \overrightarrow{\left(\frac{63}{25}, \frac{84}{25}\right)}$

$$= \overrightarrow{\left(\frac{-25}{25} - \frac{63}{25}, \frac{150}{25} - \frac{84}{25}\right)} = \overrightarrow{\left(\frac{-88}{25}, \frac{66}{25}\right)}$$

<u>CHECK</u>: $\vec{v} \cdot \left(\vec{u} - \text{proj}_{\vec{v}}\vec{u}\right)$

$$= \overrightarrow{\left(\frac{3}{5}, \frac{4}{5}\right)} \cdot \overrightarrow{\left(\frac{-88}{25}, \frac{66}{25}\right)} = \frac{-264}{125} + \frac{264}{125} = 0$$

18.) $\vec{u} = \overrightarrow{(3,-1)}$, $\vec{v} = \overrightarrow{(3,4)}$

a.) $\text{proj}_{\vec{v}}\,\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$

$$= \frac{9-4}{9+16}\,\vec{v} = \frac{1}{5}\overrightarrow{(3,4)} = \overrightarrow{\left(\tfrac{3}{5}, \tfrac{4}{5}\right)}$$

b.) $\vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \overrightarrow{(3,-1)} - \overrightarrow{\left(\tfrac{3}{5}, \tfrac{4}{5}\right)}$

$$= \overrightarrow{\left(\tfrac{15}{5} - \tfrac{3}{5}, \tfrac{-5}{5} - \tfrac{4}{5}\right)} = \overrightarrow{\left(\tfrac{12}{5}, \tfrac{-9}{5}\right)}$$

CHECK: $\vec{v}\cdot\left(\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}\right)$

$$= \overrightarrow{(3,4)}\cdot\overrightarrow{\left(\tfrac{12}{5}, \tfrac{-9}{5}\right)} = \frac{36}{5} + \frac{-36}{5} = 0$$

21.) $\vec{u} = \overrightarrow{(1,0,3)}$, $\vec{v}_1 = \overrightarrow{(1,-2,1)}$, $\vec{v}_2 = \overrightarrow{(2,1,0)}$

$W = \text{span}\{\vec{v}_1, \vec{v}_2\}$ since $\vec{v}_1 \perp \vec{v}_2$, $\vec{v}_1\cdot\vec{v}_2 = 0$

a.) $\text{proj}_W\,\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \left(\dfrac{\vec{u}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2$

$$= \frac{1+3}{1+4+1}\,\vec{v}_1 + \frac{2}{4+1}\,\vec{v}_2 = \frac{2}{3}\overrightarrow{(1,-2,1)} + \frac{2}{5}\overrightarrow{(2,1,0)}$$

$$= \overrightarrow{\left(\tfrac{2}{3}, \tfrac{-4}{3}, \tfrac{2}{3}\right)} + \overrightarrow{\left(\tfrac{4}{5}, \tfrac{2}{5}, 0\right)}$$

$$= \overrightarrow{\left(\tfrac{10}{15} + \tfrac{12}{15}, \tfrac{-20}{15} + \tfrac{6}{15}, \tfrac{2}{3} + 0\right)} = \overrightarrow{\left(\tfrac{22}{15}, \tfrac{-14}{15}, \tfrac{2}{3}\right)}$$

b.) $\vec{u} - proj_W \vec{u} = \overrightarrow{(1,0,3)} - \overrightarrow{\left(\frac{22}{15}, \frac{-14}{15}, \frac{2}{3}\right)}$

$= \overrightarrow{\left(\frac{15}{15} - \frac{22}{15}, \frac{14}{15}, \frac{9}{3} - \frac{2}{3}\right)} = \overrightarrow{\left(\frac{-7}{15}, \frac{14}{15}, \frac{7}{3}\right)}$

CHECK: $\vec{v}_1 \cdot (\vec{u} - proj_W \vec{u})$

$= \overrightarrow{(1,-2,1)} \cdot \overrightarrow{\left(\frac{-7}{15}, \frac{14}{15}, \frac{35}{15}\right)}$

$= \frac{-7}{15} + \frac{-28}{15} + \frac{35}{15} = 0$,

$\vec{v}_2 \cdot (\vec{u} - proj_W \vec{u})$

$= \overrightarrow{(2,1,0)} \cdot \overrightarrow{\left(\frac{-7}{15}, \frac{14}{5}, \frac{7}{3}\right)} = \frac{-14}{15} + \frac{14}{5} = 0$

22.) $\vec{u} = \overrightarrow{(1,0,2)}$, $\vec{v}_1 = \overrightarrow{(3,1,2)}$, $\vec{v}_2 = \overrightarrow{(-1,1,1)}$,

$W = span\{\vec{v}_1, \vec{v}_2\}$ $\leftarrow$ $\vec{v}_1 \perp \vec{v}_2$ since $\vec{v}_1 \cdot \vec{v}_2 = 0$

a.) $proj_W \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{u} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2$

$= \frac{3+4}{9+1+4}\vec{v}_1 + \frac{-1+2}{1+1+1}\vec{v}_2$

$= \frac{1}{2}\overrightarrow{(3,1,2)} + \frac{1}{3}\overrightarrow{(-1,1,1)} = \overrightarrow{\left(\frac{3}{2}, \frac{1}{2}, 1\right)} + \overrightarrow{\left(\frac{-1}{3}, \frac{1}{3}, \frac{1}{3}\right)}$

$= \overrightarrow{\left(\frac{9}{6} + \frac{-2}{6}, \frac{3}{6} + \frac{2}{6}, \frac{3}{3} + \frac{1}{3}\right)} = \overrightarrow{\left(\frac{7}{6}, \frac{5}{6}, \frac{4}{3}\right)}$

b.) $\vec{u} - \text{proj}_W \vec{u} = \overrightarrow{(1,0,2)} - \overrightarrow{\left(\frac{7}{6}, \frac{5}{6}, \frac{4}{3}\right)}$

$= \overrightarrow{\left(\frac{6}{6} - \frac{7}{6}, -\frac{5}{6}, \frac{6}{3} - \frac{4}{3}\right)} = \overrightarrow{\left(\frac{-1}{6}, \frac{-5}{6}, \frac{2}{3}\right)}$

<u>CHECK</u>: $\vec{v}_1 \cdot (\vec{u} - \text{proj}_W \vec{u})$

$= \overrightarrow{(3,1,2)} \cdot \overrightarrow{\left(\frac{-1}{6}, \frac{-5}{6}, \frac{4}{6}\right)} = \frac{-3}{6} + \frac{-5}{6} + \frac{8}{6} = 0,$

$\vec{v}_2 \cdot (\vec{u} - \text{proj}_W \vec{u})$

$= \overrightarrow{(-1,1,1)} \cdot \overrightarrow{\left(-\frac{1}{6}, \frac{-5}{6}, \frac{4}{6}\right)} = \frac{1}{6} + \frac{-5}{6} + \frac{4}{6} = 0$

23.) $\vec{u} = \overrightarrow{(1,2,0,-2)}$, $\vec{v}_1 = \overrightarrow{(1,1,1,1)}$, $\vec{v}_2 = \overrightarrow{(1,1,-1,-1)}$,

$W = \text{span}\{\vec{v}_1, \vec{v}_2\}$ ($\vec{v}_1 \perp \vec{v}_2$ since $\vec{v}_1 \cdot \vec{v}_2 = 0$)

then

$\text{proj}_W \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1 + \left(\frac{\vec{u} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right) \vec{v}_2$

$= \frac{1+2-2}{1+1+1+1} \vec{v}_1 + \frac{1+2+2}{1+1+1+1} \vec{v}_2$

$= \frac{1}{4} \overrightarrow{(1,1,1,1)} + \frac{5}{4} \overrightarrow{(1,1,-1,-1)}$

$= \overrightarrow{\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)} + \overrightarrow{\left(\frac{5}{4}, \frac{5}{4}, \frac{-5}{4}, \frac{-5}{4}\right)}$

$= \overrightarrow{\left(\frac{3}{2}, \frac{3}{2}, -1, -1\right)}$

26.) $\vec{u} = \overrightarrow{(1,2,0,-1)}$, $\vec{v}_1 = \overrightarrow{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}$,

$\vec{v}_2 = \overrightarrow{\left(\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2}\right)}$, $\vec{v}_3 = \overrightarrow{\left(\frac{1}{2},\frac{-1}{2},\frac{1}{2},\frac{-1}{2}\right)}$;

$\vec{v}_1 \cdot \vec{v}_2 = 0$, $\vec{v}_1 \cdot \vec{v}_3 = 0$, $\vec{v}_2 \cdot \vec{v}_3 = 0$,

$\|\vec{v}_1\| = \|\vec{v}_2\| = \|\vec{v}_3\| = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = 1$,

so $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are ORTHONORMAL ;

$W = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, then

$$\text{proj}_W \vec{u} = (\vec{u}\cdot v_1)\vec{v}_1 + (\vec{u}\cdot\vec{v}_2)\vec{v}_2 + (\vec{u}\cdot\vec{v}_3)\vec{v}_3$$

$$= \left(\frac{1}{2}+\frac{2}{2}-\frac{1}{2}\right)\vec{v}_1 + \left(\frac{1}{2}+\frac{2}{2}+\frac{1}{2}\right)\vec{v}_2 + \left(\frac{1}{2}-\frac{2}{2}+\frac{1}{2}\right)\vec{v}_3$$

$$= (1)\overrightarrow{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} + (2)\overrightarrow{\left(\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2}\right)} + (0)\overrightarrow{\left(\frac{1}{2},\frac{-1}{2},\frac{1}{2},\frac{-1}{2}\right)}$$

$$= \overrightarrow{\left(\frac{1}{2}+\frac{2}{2}, \frac{1}{2}+\frac{2}{2}, \frac{1}{2}-\frac{2}{2}, \frac{1}{2}-\frac{2}{2}\right)} = \overrightarrow{\left(\frac{3}{2},\frac{3}{2},\frac{-1}{2},\frac{-1}{2}\right)}$$

27.) $\vec{u}_1 = \overrightarrow{(1,-3)}$, $\vec{u}_2 = \overrightarrow{(2,2)}$

i.) $\vec{v}_1 = \vec{u}_1 = \overrightarrow{(1,-3)}$

ii.) $\vec{v}_2 = \vec{u}_2 - \left(\dfrac{\vec{u}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1$

$$= \vec{u}_2 - \frac{2-6}{1+9}\vec{v}_1 = \overrightarrow{(2,2)} + \frac{2}{5}\overrightarrow{(1,-3)}$$

$$= \overrightarrow{\left(\frac{10}{5}+\frac{2}{5}, \frac{10}{5}+\frac{-6}{5}\right)} = \overrightarrow{\left(\frac{12}{5},\frac{4}{5}\right)} \Rightarrow$$

now make unit vectors:

$$\frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{1+9}}\overrightarrow{(1,-3)} = \boxed{\overrightarrow{\left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right)}};$$

$$\frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{\sqrt{\frac{144}{25}+\frac{16}{25}}}\overrightarrow{\left(\frac{12}{5}, \frac{4}{5}\right)}$$

$$= \frac{1}{\sqrt{\frac{160}{25}}}\overrightarrow{\left(\frac{12}{5}, \frac{4}{5}\right)} = \frac{5}{4\sqrt{10}}\overrightarrow{\left(\frac{12}{5}, \frac{4}{5}\right)}$$

$$= \boxed{\overrightarrow{\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)}}$$

28.) $\vec{u}_1 = \overrightarrow{(1,0)}$, $\vec{u}_2 = \overrightarrow{(3,-5)}$

i.) $\vec{v}_1 = \vec{u}_1 = \overrightarrow{(1,0)}$

ii) $\vec{v}_2 = \vec{u}_2 - \left(\frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1$

$= \vec{u}_2 - \frac{3+0}{1+0}\vec{v}_1 = \overrightarrow{(3,-5)} - 3\overrightarrow{(1,0)}$

$= \overrightarrow{(0,-5)} \Rightarrow$ now make unit vectors: $\frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \boxed{\overrightarrow{(1,0)}}$;

$$\frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{5}\overrightarrow{(0,-5)} = \boxed{\overrightarrow{(0,-1)}}$$

29.) $\vec{u_1} = \overrightarrow{(1,1,1)}$, $\vec{u_2} = \overrightarrow{(-1,1,0)}$, $\vec{u_3} = \overrightarrow{(1,2,1)}$

i.) $\vec{v_1} = \vec{u_1} = \overrightarrow{(1,1,1)}$

ii.) $\vec{v_2} = \vec{u_2} - \left(\frac{\vec{u_2}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\right)\vec{v_1}$

$= \vec{u_2} - \frac{-1+1}{1+1+1}\vec{v_1} = \overrightarrow{(-1,1,0)} - (0)\overrightarrow{(1,1,1)}$

$= \overrightarrow{(-1,1,0)}$

iii) $\vec{v_3} = u_3 - \left(\frac{\vec{u_3}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\right)\vec{v_2} - \left(\frac{\vec{u_3}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\right)\vec{v_1}$

$= \vec{u_3} - \frac{-1+2}{1+1}\vec{v_2} - \frac{1+2+1}{1+1+1}\vec{v_1}$

$= \overrightarrow{(1,2,1)} - \frac{1}{2}\overrightarrow{(-1,1,0)} - \frac{4}{3}\overrightarrow{(1,1,1)}$

$= \overrightarrow{\left(\frac{6}{6}+\frac{3}{6}-\frac{8}{6}, \frac{12}{6}-\frac{3}{6}-\frac{8}{6}, \frac{6}{6}-\frac{8}{6}\right)}$

$= \overrightarrow{\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)}$ $\Rightarrow$ now make unit vectors:

$\frac{1}{\|\vec{v_1}\|}\vec{v_1} = \frac{1}{\sqrt{1+1+1}}\overrightarrow{(1,1,1)} = \boxed{\overrightarrow{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}}$,

$\frac{1}{\|\vec{v_2}\|}\vec{v_2} = \frac{1}{\sqrt{1+1}}\overrightarrow{(-1,1,0)} = \boxed{\overrightarrow{\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}}$

$$\frac{1}{\|\vec{v_3}\|}\vec{v_3} = \frac{1}{\sqrt{\frac{1}{36}+\frac{1}{36}+\frac{4}{36}}}\overrightarrow{\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)}$$

$$= \sqrt{\frac{36}{6}}\ \overrightarrow{\left(\frac{1}{6}, \frac{1}{6}, \frac{-2}{6}\right)} = \boxed{\overrightarrow{\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)}}$$

32.) $\vec{u_1} = \overrightarrow{(0,1,2)}$, $\vec{u_2} = \overrightarrow{(-1,0,1)}$, $\vec{u_3} = \overrightarrow{(-1,1,3)}$;

first find basis for subspace:

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so basis is $\{\vec{u_1}, \vec{u_2}\}$:

i.) $\vec{v_1} = \vec{u_1} = \overrightarrow{(0,1,2)}$

ii.) $\vec{v_2} = \vec{u_2} - \left(\dfrac{\vec{u_2}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\right)\vec{v_1}$

$$= \vec{u_2} - \frac{2}{1+4}\vec{v_1} = \overrightarrow{(-1,0,1)} - \frac{2}{5}\overrightarrow{(0,1,2)}$$

$= \overrightarrow{\left(-1, \frac{-2}{5}, \frac{1}{5}\right)} \Rightarrow$ now make

unit vectors:

$$\frac{1}{\|\vec{v_1}\|}\vec{v_1} = \frac{1}{\sqrt{1+4}}\overrightarrow{(0,1,2)} = \boxed{\overrightarrow{\left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)}},$$

$$\frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{\sqrt{1+\frac{4}{25}+\frac{1}{25}}}\left(\overrightarrow{-1, \frac{-2}{5}, \frac{1}{5}}\right)$$

$$= \frac{1}{\sqrt{\frac{30}{25}}}\left(\overrightarrow{-1, \frac{-2}{5}, \frac{1}{5}}\right) = \frac{5}{\sqrt{30}}\left(\overrightarrow{-1, \frac{-2}{5}, \frac{1}{5}}\right)$$

$$= \boxed{\left(\overrightarrow{-\frac{5}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}}\right)}$$

33.) $\vec{u} = (\overrightarrow{1, 2, 0, -2})$, $\vec{v}_1 = (\overrightarrow{1,1,1,1})$, $\vec{v}_2 = (\overrightarrow{1,1,-1,-1})$;

$\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for $W = \text{span}\{\vec{v}_1, \vec{v}_2\}$; if $\vec{u} = \vec{w}_1 + \vec{w}$, where $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^{\perp}$: then

$$\vec{w}_1 = \text{proj}_W \vec{u}$$

$$= \left(\frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{u}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2$$

$$= \frac{1+2-2}{1+1+1+1}\vec{v}_1 + \frac{1+2+2}{1+1+1+1}\vec{v}_2$$

$$= \frac{1}{4}(\overrightarrow{1,1,1,1}) + \frac{5}{4}(\overrightarrow{1,1,-1,-1}) = \left(\overrightarrow{\frac{3}{2}, \frac{3}{2}, -1, -1}\right);$$

$$\vec{w_2} = \vec{u} - \vec{w_1}$$

$$= \overrightarrow{(1,2,0,-2)} - \overrightarrow{(\tfrac{3}{2},\tfrac{3}{2},-1,-1)}$$

$$= \overrightarrow{(-\tfrac{1}{2},\tfrac{1}{2},1,-1)} \in W^{\perp} \text{ since}$$

$\vec{w_2} \cdot \vec{v_1} = 0$ and $\vec{w_2} \cdot \vec{v_2} = 0$.

(*) 43.) $\vec{u}_1 = 1$, $\vec{u}_2 = x$, $\vec{u}_3 = x^2$, define

$$\overrightarrow{p(x)} \cdot \overrightarrow{q(x)} = \int_0^1 p(x)q(x)\,dx \; :$$

i.) $\vec{v}_1 = \vec{u}_1 = 1$

ii.) $\vec{v}_2 = \vec{u}_2 - \left(\dfrac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1$

$$= \vec{u}_2 - \frac{\int_0^1 x\,dx}{\int_0^1 1\,dx}\vec{v}_1$$

$$= \vec{u}_2 - \frac{\frac{1}{2}x^2\big|_0^1}{x\big|_0^1}\vec{v}_1$$

$$= \vec{u}_2 - \frac{1/2}{1}\vec{v}_1 = x - \frac{1}{2}$$

iii.) $\vec{v}_3 = \vec{u}_3 - \left(\dfrac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 - \left(\dfrac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1$

$$= \vec{u}_3 - \frac{\int_0^1 x^2(1-x)\,dx}{\int_0^1 (x-\frac{1}{2})^2 dx}\vec{v}_2 - \frac{\int_0^1 x^2\,dx}{\int_0^1 1\,dx}\vec{v}_1$$

$$= \vec{u}_3 - \frac{\int_0^1 (x^2 - x^3)\,dx}{\int_0^1 (x^2 - x + \frac{1}{4})\,dx}\vec{v}_2 - \frac{\frac{1}{3}x^3\Big|_0^1}{x\Big|_0^1}\vec{v}_1$$

$$= \vec{u}_3 - \frac{\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1}{\left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x\right)\Big|_0^1}\vec{v}_2 - \frac{\frac{1}{3}}{1}v_1$$

$$= \vec{u}_3 - \frac{1/12}{1/12}\vec{v}_2 - \frac{1}{3}\vec{v}_1$$

$$= x^2 - (x - \tfrac{1}{2}) - \tfrac{1}{3}(1)$$

$= x^2 - x + \frac{1}{6}$ $\Rightarrow$ now make unit vectors:

$$\frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{\int_0^1 1\,dx}}(1) = \boxed{1};$$

$$\frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{\sqrt{\int_0^1 (x-\frac{1}{2})^2 dx}}\vec{v}_2$$

$$= \frac{1}{\sqrt{1/12}}\vec{v}_2 = \boxed{2\sqrt{3}\left(x - \tfrac{1}{2}\right)};$$

$$\frac{1}{\|\vec{v}_3\|}\vec{v}_3 = \frac{1}{\sqrt{\int_0^1 (x^2 - x + \frac{1}{6})^2\,dx}}\ \vec{v}_3$$

$$= \frac{1}{\sqrt{\int_0^1 (x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36})\,dx}}\ \vec{v}_3$$

$$= \frac{1}{\sqrt{\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}}}\ \vec{v}_3$$

$$= \frac{1}{\sqrt{\frac{36}{180} - \frac{90}{180} + \frac{80}{180} - \frac{30}{180} + \frac{5}{180}}}\ \vec{v}_3$$

$$= \frac{1}{\sqrt{1/180}}\ \vec{v}_3 = \boxed{6\sqrt{5}\,(x^2 - x + \tfrac{1}{6})}$$

TRUE / FALSE

(c) T (e) F

55.) assume W is subspace of vector space V (finite dimension). Define

$T : V \to W$ by $T(\vec{v}) = \mathrm{proj}_W \vec{v}$.

a.) Show T is a linear transformation, i.e., show

i.) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in V$

ii.) $T(\alpha \vec{v}) = \alpha T(\vec{v})$ for all $\vec{v} \in V$, constants α

i.) Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be an orthonormal basis for W. Then for $\vec{u}, \vec{v} \in V$ we have

$$T(\vec{u}+\vec{v}) = \mathrm{proj}_W (\vec{u}+\vec{v})$$

$$= \left(\frac{(\vec{u}+\vec{v}) \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + \dots + \left(\frac{(\vec{u}+\vec{v}) \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}\right)\vec{v}_k$$

$$= \frac{(\vec{u}\cdot\vec{v}_1) + (\vec{v}\cdot\vec{v}_1)}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \dots$$

$$+ \frac{(\vec{u}\cdot\vec{v}_k) + (\vec{v}\cdot\vec{v}_k)}{\vec{v}_k \cdot \vec{v}_k}\vec{v}_k$$

$$= \left(\frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \cdots + \left(\frac{\vec{u}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\right)\vec{v}_k$$

$$+ \left(\frac{\vec{v}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \cdots + \left(\frac{\vec{v}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\right)\vec{v}_k$$

$$= \operatorname{proj}_W \vec{u} + \operatorname{proj}_W \vec{v}$$

ii.) For $\vec{v} \in V$ and constant α we have

$$T(\alpha\vec{v}) = \operatorname{proj}_W (\alpha\vec{v})$$

$$= \left(\frac{\alpha\vec{v}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \cdots + \left(\frac{\alpha\vec{v}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\right)\vec{v}_k$$

$$= \alpha\left(\frac{\vec{v}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \cdots + \alpha\left(\frac{\vec{v}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\right)\vec{v}_k$$

$$= \alpha\left[\left(\frac{\vec{v}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \cdots + \left(\frac{\vec{v}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\right)\vec{v}_k\right]$$

$$= \alpha \operatorname{proj}_W \vec{v}$$

b.) Kernel of $T = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$;

Show Kernel of $T = W^{\perp}$, where

$W^{\perp} = \{\vec{v} \in V \mid \vec{v} \cdot \vec{w} = 0$ for all $w \in W\}$:

($\supseteq$) Show $W^{\perp} \subseteq$ Kernel of T:

If $\vec{v} \in W^{\perp} \Rightarrow \vec{v} \cdot \vec{w} = 0$ for all $\vec{w} \in W$

$\Rightarrow \vec{v} \cdot \vec{v}_1 = 0, \vec{v} \cdot \vec{v}_2 = 0, \ldots,$ and $\vec{v} \cdot \vec{v}_k = 0$ since $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in W$

$\Rightarrow T(\vec{v}) = \text{proj}_W \vec{v}$

$$= \left(\frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + \cdots + \left(\frac{\vec{v} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}\right)\vec{v}_k$$

$$= (0)\vec{v}_1 + \cdots + (0)\vec{v}_k = \vec{0} \Rightarrow$$

$\vec{v} \in$ Kernel of T

($\subseteq$) Show Kernel of $T \subseteq W^{\perp}$.

First extend the basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ for W to an orthonormal basis

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{u}_{k+1}, \vec{u}_{k+2}, \ldots, \vec{u}_r\}$,

where $\vec{u}_{k+1}, \vec{u}_{k+2}, \ldots, \vec{u}_r \in W^\perp$.

Now let $\vec{v} \in$ Kernel of $T \Rightarrow$

$$\vec{v} = \alpha_1 \vec{v}_1 + \cdots + \alpha_k \vec{v}_k + \alpha_{k+1} \vec{u}_{k+1} + \cdots + \alpha_r \vec{u}_r \text{ and}$$

$$T(\vec{v}) = \vec{0} \Rightarrow$$

$$T(\vec{v}) = \text{proj}_W \vec{v}$$

$$= \left(\frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1 + \cdots + \left(\frac{\vec{v} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}\right) \vec{v}_k$$

$$= \frac{(\alpha_1 \vec{v}_1 + \cdots + \alpha_k \vec{v}_k + \alpha_{k+1} \vec{u}_{k+1} + \cdots + \alpha_r \vec{u}_r) \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$+ \cdots + \frac{(\alpha_1 \vec{v}_1 + \cdots + \alpha_k \vec{v}_k + \alpha_{k+1} \vec{u}_{k+1} + \cdots + \alpha_r \vec{u}_r] \cdot v_k}{\vec{v}_k \cdot \vec{v}_k} \vec{v}_k$$

$$= \frac{\alpha_1 (\vec{v}_1 \cdot \vec{v}_1) + \cdots + 0 + 0 + \cdots + 0}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \cdots$$

$$+ \frac{0 + \cdots + \alpha_k (\vec{v}_k \cdot \vec{v}_k) + 0 + \cdots + 0}{\vec{v}_k \cdot \vec{v}_k} \vec{v}_k$$

$= \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \cdots + \alpha_k \vec{v_k} = \vec{0}$

$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$ (since $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ are linearly independent)

$\Rightarrow \vec{v} = \alpha_{k+1} \vec{u_{k+1}} + \cdots + \alpha_r \vec{u_r} \in W^{\perp}$.

This completes the proof.

Range of $T = \{\vec{w} \in W \mid T(\vec{v}) = \vec{w}$ for some $\vec{v} \in V\}$;

Show Range of $T = W$:

($\subseteq$) Show Range of $T \subseteq W$:

Let $\vec{w} \in$ Range of T, then $\vec{w} \in W$.

($\supseteq$) Show $W \subseteq$ Range of T:

Let $\vec{w} \in W$, then

$T(\vec{w}) = \text{proj}_W \vec{w} = \vec{w} \Rightarrow$

$\vec{w} \in$ Range of T.

This completes the proof.