Section 5.1

1.) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so

eigenvalue is $\lambda = -1$

2.) $\begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = (4) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so

eigenvalue is $\lambda = 4$

4.) $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (0) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so

eigenvalue is $\lambda = 0$

5.) a.) $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix}$

$= (1-\lambda)(3-\lambda) - 8$

$= \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0 \Rightarrow$

$\lambda_1 = 5, \ \lambda_2 = -1$

<u>For $\lambda_1 = 5$</u>: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow x_1 - x_2 = 0$, so

let $x_2 = t$ any # $\Rightarrow x_1 = t \Rightarrow$
eigenvector is

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so

$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is basis for eigenspace;

<u>For $\lambda_2 = -1$:</u> Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 + 2x_2 = 0$,

so let $x_2 = t$ any # $\Rightarrow x_1 = -2t \Rightarrow$ eigenvector is

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, so

$B = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is basis for eigenspace.

b.) $\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -7 \\ 1 & 2-\lambda \end{vmatrix}$

$= (-2-\lambda)(2-\lambda) - (-7)$

$= \lambda^2 + 3 = 0$ (NO REAL SOLUTIONS)

c.) $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \Rightarrow$

$\lambda = 1$;

<u>For $\lambda=1$</u>: Solve $(A-\lambda I)\vec{x} = \vec{0}$:

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow (0)x_1 + (0)x_2 = 0 \Rightarrow$

$x_1 = t$ any #, $x_2 = r$ any # $\Rightarrow$

eigenvector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ r \end{bmatrix}$

$= \begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ r \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so

basis for eigenspace is

$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

d.) $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$

$\Rightarrow \lambda = 1$;

<u>For $\lambda = 1$</u>: Solve $(A-\lambda I)\vec{x} = \vec{0}$:

$\left[\begin{array}{cc|c} 0 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow (0)x_1 + x_2 = 0$

$\Rightarrow x_2 = 0$ and $x_1 = t$ any # $\Rightarrow$

eigenvector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is basis for eigenspace

6.) a.) $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$

$= (2-\lambda)(2-\lambda) - 1$

$= \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0 \Rightarrow$

$\lambda_1 = 3, \quad \lambda_2 = 1 \quad ;$

<u>For $\lambda_1 = 3$</u>: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow x_1 - x_2 = 0 \Rightarrow$

$x_2 = t$ any # $\Rightarrow x_1 = t \Rightarrow$

eigenvector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

so basis for eigenspace is

$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

<u>For $\lambda_2 = 1$</u>: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 + x_2 = 0 \Rightarrow$

$x_2 = t$ any # $\Rightarrow x_1 = -t$, so

eigenvector is $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

and basis for eigenspace is

$B = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

7.) $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix}$

$= (4-\lambda)\left[(1-\lambda)^2 - 0\right] - (0)\left[-2(1-\lambda) - 0\right]$

$\quad + (1)\left[0 - {}^{-}2(1-\lambda)\right]$

$= (4-\lambda)(1-\lambda)^2 + 2(1-\lambda)$

$= (1-\lambda)\left[\lambda^2 - 5\lambda + 4 + 2\right]$

$= (1-\lambda)(\lambda - 5\lambda + 6)$

$= (1-\lambda)(\lambda-2)(\lambda-3) = 0 \Rightarrow$

$\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3$

<u>For $\lambda_1 = 1$</u>: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow$$

$x_3 = 0$, $x_1 = 0$, and $(0)\,x_2 = 0 \Rightarrow$

$x_2 = t$ any #, so eigenvector

is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and

basis for eigenspace is $B = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$;

For $\lambda_2 = 2$: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow$$

$$\begin{cases} 2x_1 + x_3 = 0 \\ -x_2 + x_3 = 0 \end{cases} \Rightarrow \quad x_3 = t \text{ any } \# \Rightarrow$$

$x_1 = -\frac{1}{2}t$ and $x_2 = t \Rightarrow$

eigenvector is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t \\ t \\ t \end{bmatrix}$

$= \frac{t}{2}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, so basis for eigenspace

is $B = \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$;

<u>For $\lambda_3 = 3$</u>: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{cases} x_1 + x_3 = 0 \\ -x_2 + x_3 = 0 \end{cases} \Rightarrow$$

$x_3 = t$ any # $\Rightarrow x_1 = -t$, $x_2 = t$,

so eigenvector is

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, and basis

for eigenspace is $B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

10.) $\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$

$= -\lambda(\lambda^2-1) - 1(-\lambda-1) + 1(1+\lambda)$

$= -\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda$

$= -\lambda^3 + 3\lambda + 2$

$= -(\lambda^3 - 3\lambda - 2)$

$= -(\lambda+1)(\lambda^2 - \lambda - 2)$

$= -(\lambda+1)(\lambda-2)(\lambda+1)$

$= -(\lambda+1)^2(\lambda-2) = 0 \Rightarrow$

$\lambda_1 = -1, \lambda_2 = 2$

For $\lambda_1 = -1$: Solve $(A-\lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow$$

$x_1 + x_2 + x_3 = 0 \Rightarrow x_3 = t$ any #,

$x_2 = r$ any # $\Rightarrow x_1 = -t - r \Rightarrow$

eigenvector is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t-r \\ r \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} -r \\ r \\ 0 \end{bmatrix}$$

$$= t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$ so basis

for eigenspace is

$$B = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \quad ;$$

<u>For $\lambda_2 = 2$</u>: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & -3 & 3 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$$\begin{array}{c} \quad\; x_1 \quad\; x_2 \quad\; x_3 \\ \sim \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{cases} x_1 - x_3 = 0 \\ -x_2 + x_3 = 0 \end{cases} \Rightarrow$$

$x_3 = t$ any # $\Rightarrow$ $x_1 = t$, $x_2 = t$,

so eigenvector is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$ and basis

for eigenspace is $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

12.) $$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(-5-\lambda)(4-\lambda) - (-18)\right]$$
$$-(-3)\left[3(4-\lambda) - 18\right]$$
$$+ 3\left[-18 - 6(-5-\lambda)\right]$$

$$= (1-\lambda)\left[\lambda^2 + \lambda - 2\right]$$
$$+ 3\left[-3\lambda - 6\right] + 3\left[6\lambda + 12\right]$$

$$= (1-\lambda)(\lambda-1)(\lambda+2)$$
$$+ -9\lambda - 18 + 18\lambda + 36$$
$$= (1-\lambda)(\lambda-1)(\lambda+2) + 9(\lambda+2)$$
$$= (\lambda+2)\left[-\lambda^2 + 2\lambda - 1 + 9\right]$$
$$= (\lambda+2)(-1)(\lambda - 2\lambda - 8)$$

$= -(\lambda+2)(\lambda-4)(\lambda+2)$

$= -(\lambda+2)^2(\lambda-4) = 0 \Rightarrow$

$\lambda_1 = -2, \quad \lambda_2 = 4$

<u>For $\lambda_1 = -2$</u>: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow$$

$x_1 - x_2 + x_3 = 0 \Rightarrow x_3 = t$ any #,

$x_2 = r$ any # $\Rightarrow x_1 = r - t \Rightarrow$

eigenvector is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r - t \\ r \\ t \end{bmatrix}$

$= \begin{bmatrix} r \\ r \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, so

basis for eigenspace is

$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$;

For $\lambda_2 = 4$: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & \tfrac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -\tfrac{1}{2} & 0 \\ 0 & 1 & -\tfrac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow$$

$$\begin{cases} x_1 - \frac{1}{2}x_3 = 0 \\ x_2 - \frac{1}{2}x_3 = 0 \end{cases} \Rightarrow \quad x_3 = t \text{ any \#} \Rightarrow$$

$x_1 = \frac{1}{2}t$, $x_2 = \frac{1}{2}t \Rightarrow$ eigenvector

is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ \frac{1}{2}t \\ t \end{bmatrix} = \frac{1}{2}t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, so

basis for eigenspace is $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$.

24.) $\det(A-\lambda I) = p(\lambda) \Rightarrow$ (Let $\lambda=0$)

$\det(A) = \det(A-(0)I) = p(0)$

a.) $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5 \Rightarrow$

$\det(A) = p(0) = 5$

b.) $p(\lambda) = \lambda^4 - \lambda^3 + 7 \Rightarrow$

$\det(A) = p(0) = 7$

25.) $p(\lambda) = (\lambda-1)(\lambda-3)^2(\lambda-4)^3$

a.) A is 6×6 since degree of p is 6

b.) $\det(A) = p(0) = (-1)(-3)^2(-4)^3 \neq 0$

so A is invertible

c.) A has 3 eigenspaces since has 3 eigenvalues (1, 3, and 4)

27.) Find matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, so

that

$$A\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

$$A\begin{bmatrix}1\\1\\0\end{bmatrix}=(-1)\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\-1\\0\end{bmatrix},$$

$$A\begin{bmatrix}1\\-1\\0\end{bmatrix}=(0)\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad\Rightarrow$$

$$A\begin{bmatrix}1&1&1\\-1&1&-1\\1&0&0\end{bmatrix}=\begin{bmatrix}1&-1&0\\-1&-1&0\\1&0&0\end{bmatrix}\quad\Rightarrow$$

$$A=\begin{bmatrix}1&-1&0\\-1&-1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&1&1\\-1&1&-1\\1&0&0\end{bmatrix}^{-1};$$

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\-1&1&-1&0&1&0\\1&0&0&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&2&0&1&1&0\\0&1&-1&0&1&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1&0&2&1&-1&-1\\0&0&2&1&-1&-2\\0&1&-1&0&1&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&2&1&-1&-1\\0&0&1&\frac{1}{2}&-\frac{1}{2}&-1\\0&1&-1&0&1&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1&0&0&0&0&1\\0&0&1&\frac{1}{2}&-\frac{1}{2}&-1\\0&1&0&\frac{1}{2}&\frac{1}{2}&0\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&0&0&1\\0&1&0&\frac{1}{2}&\frac{1}{2}&0\\0&0&1&\frac{1}{2}&-\frac{1}{2}&-1\end{array}\right]\quad\Rightarrow$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & -1 \end{bmatrix} \Rightarrow$$

$$A = \begin{bmatrix} -1/2 & -1/2 & 1 \\ -1/2 & -1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

33.) If λ is an eigenvalue for an invertible matrix A and $\vec{x}$ is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue for A^{-1} and $\vec{x}$ is a corresponding eigenvector:

A is invertible $\Rightarrow \det(A) \neq 0$

$\Rightarrow$ none of its eigenvalues can be equal to zero (by Theorem D from class); and

$$A\vec{x} = \lambda \vec{x} \Rightarrow A^{-1}(A\vec{x}) = A^{-1}(\lambda \vec{x})$$

$$\Rightarrow (A^{-1}A)\vec{x} = \lambda (A^{-1}\vec{x})$$

$$\Rightarrow I\vec{x} = \lambda (A^{-1}\vec{x})$$

$$\Rightarrow A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x} \quad (\text{since } \lambda \neq 0) \Rightarrow$$

$1/\lambda$ is an eigenvalue for A^{-1} with eigenvector $\vec{x}$.

Q.E.D

34.) If $A\vec{x} = \lambda\vec{x}$, then $(A-sI)\vec{x} = (\lambda-s)\vec{x}$:

$$(A-sI)\vec{x} = A\vec{x} - sI\vec{x}$$
$$= \lambda\vec{x} - s\vec{x}$$
$$= (\lambda - s)\vec{x} \Rightarrow \lambda - s \text{ is}$$

an eigenvalue for $A-sI$ with eigenvector $\vec{x}$.

Q.E.D.

38.) a.) If A is square, then A and A^T have the same eigenvalues:

$$p(\lambda) = \det(A - \lambda I)$$
$$= \det(A-\lambda I)^T$$
$$(\text{since } \det(B) = \det(B^T))$$
$$= \det(A^T - (\lambda I)^T)$$
$$= \det(A^T - \lambda I) \Rightarrow$$

A and A^T have the same

characteristic polynomial $\Rightarrow$
A and A^T have the same
eigenvalues.

Q.E.D.

6.) Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow$

$\det(A - \lambda I) = \begin{bmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = 0$

$\Rightarrow \lambda = 1$

For $\lambda = 1$: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \Rightarrow (1)x_1 + (0)x_2 = 0 \Rightarrow$

$x_1 = 0$ and $x_2 = t$ any # $\Rightarrow$

eigenvector is

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so basis

for eigenspace is $B = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$;

then $A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow$

$\det(A^T - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \Rightarrow$

$\lambda = 1$

For $\lambda = 1$: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow (0)x_1 + (1)x_2 = 0 \Rightarrow$$

$x_1 = t$ any # and $x_2 = 0 \Rightarrow$
eigenvector is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$ so basis

for eigenspace is $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

39.) Assume that α and β are distinct eigenvalues for A with eigenvectors $\vec{X}$ and $\vec{Y}$ resp. Let V be the eigenspace for α, and W be the eigenspace for β. Show $V \wedge W = \{\vec{0}\}$:

Assume $\vec{Z} \in V \wedge W \Rightarrow$

$\vec{z} \in V \Rightarrow A\vec{z} = \alpha \vec{z}$ and

$\vec{z} \in W \Rightarrow A\vec{z} = \beta \vec{z}$. Then

$$\vec{0} = A\vec{z} - A\vec{z}$$
$$= \alpha\vec{z} - \beta\vec{z}$$
$$= (\alpha - \beta)\vec{z} \quad (\text{but } \alpha \neq \beta)$$
$$\Rightarrow \vec{z} = \frac{1}{\alpha - \beta}\vec{0} = \vec{0} \Rightarrow$$

$$V \cap W = \{\vec{0}\}$$

QED.

TRUE/FALSE

(a) F (b) F (c) T (d) F

(e) F (f) F