Section 5.2

1.) $\det(A) = 2-3 = -1$, $\det(B) = -2-0 = -2$,
so A and B NOT similar since
$\det(A) \neq \det(B)$

4.) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $\text{rank}(A) = 1$,

$B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, so $\text{rank}(B) = 2$,

so A and B NOT similar since
$\text{rank}(A) \neq \text{rank}(B)$

5.) $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix}$

$= (1-\lambda)(-1-\lambda) - 0 = (\lambda - 1)(\lambda + 1) = 0 \Rightarrow$
$\lambda_1 = 1$, $\lambda_2 = -1$;

<u>For $\lambda_1 = 1$</u>: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 3 & -1 & 0 \end{array}\right] \Rightarrow 3x_1 - x_2 = 0 \Rightarrow$

let $x_1 = t$ any # $\Rightarrow x_2 = 3t \Rightarrow$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 3t \end{bmatrix} = t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

is eigenvector ;

For $\lambda_2 = -1$: solve $(A - \lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c} 2 & 0 & 0 \\ 6 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow (1)x_1 + (0)x_2 = 0 \Rightarrow$$

$x_1 = 0$ and $x_2 = t$ any # $\Rightarrow$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$ so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is

an eigenvector .

Let $\boxed{P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}} \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right] \Rightarrow P^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \Rightarrow$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D$$

8.) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix}$

$$= (1-\lambda)\left[(1-\lambda)^2 - 1\right] = (1-\lambda)\left[\lambda^2 - 2\lambda + 1 - 1\right]$$

$= (1-\lambda)\lambda(\lambda-2) = 0 \Rightarrow$

$\lambda_1 = 0, \ \lambda_2 = 1, \ \lambda_3 = 2$;

For $\lambda_1 = 0$: Solve $(A-\lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

$\Rightarrow$ let $x_3 = t$ any # $\Rightarrow x_2 = -t$, so

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

is eigenvector ;

For $\lambda_2 = 1$: Solve $(A-\lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_2 = 0, \ x_3 = 0 \text{ and} \\ (0)x_1 + 0 + 0 = 0 \Rightarrow \end{cases}$$

let $x_1 = t$ any #, so

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ is}$$

eigenvector ;

For $\lambda_3 = 2$: Solve $(A-\lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = 0 \text{ and} \\ x_2 - x_3 = 0 \end{cases}$$

$\Rightarrow$ let $x_3 = t$ any # $\Rightarrow$ $x_2 = t$, so

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ is}$$

eigenvector.

Let $P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \Rightarrow$$

$$P^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \text{ then}$$

$$P^{-1}AP = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

17.) $A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 3 \\ 2 & -1-\lambda \end{vmatrix}$

$= -\lambda(-1-\lambda) - 6 = \lambda^2 + \lambda - 6$

$= (\lambda - 2)(\lambda + 3) = 0 \Rightarrow$

$\lambda_1 = 2, \ \lambda_2 = -3$;

<u>For $\lambda_1 = 2$:</u> Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$\left[\begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & -3 & 0 \end{array}\right] \Rightarrow 2x_1 - 3x_2 = 0$

$\Rightarrow$ let $x_2 = t$ any # $\Rightarrow 2x_1 = 3t \Rightarrow$

$x_1 = \frac{3}{2}t$, so $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}t \\ t \end{bmatrix}$

$= \frac{t}{2}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, so eigenvector is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$;

For $\lambda_2 = -3$: Solve $(A - \lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right] \Rightarrow x_1 + x_2 = 0 \Rightarrow$$

let $x_2 = t$ any # $\Rightarrow x_1 = -t$, so

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix},$$ and

eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Let $P = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{cc|cc} 1 & -2 & 1 & -1 \\ 2 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -2 & 1 & -1 \\ 0 & 5 & -2 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & -2 & 1 & -1 \\ 0 & 1 & -2/5 & 3/5 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/5 & 1/5 \\ 0 & 1 & -2/5 & 3/5 \end{array}\right] \Rightarrow$$

$$P^{-1} = \begin{bmatrix} 1/5 & 1/5 \\ -2/5 & 3/5 \end{bmatrix} \Rightarrow$$

$$P^{-1}AP = \begin{bmatrix} 1/5 & 1/5 \\ -2/5 & 3/5 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & 1/5 \\ -2/5 & 3/5 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} = D \Rightarrow$$

$$P^{-1}AP = D \Rightarrow A = PDP^{-1} \Rightarrow$$

$$A^{10} = (PDP^{-1})^{10}$$

$$= PD^{10}P^{-1}$$

$$= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 \\ 0 & (-3)^{10} \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ -2/5 & 3/5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot 2^{10} & -3^{10} \\ 2^{11} & 3^{10} \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ -2/5 & 3/5 \end{bmatrix}$$

$$= \begin{bmatrix} 24234 & -34815 \\ -23210 & 35839 \end{bmatrix}$$

20.) $A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and

$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $A \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$,

$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so

$AP = PD$, where $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

Find P^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -1 & 4 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{array}\right] \Rightarrow P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix},$$

then $\boxed{A = PDP^{-1}} \Rightarrow$

$$A^{-1} = (PDP^{-1})^{-1}$$

$$= (P^{-1})^{-1} D^{-1} P^{-1} = P D^{-1} P^{-1}, \text{ i.e.,}$$

$\boxed{A^{-1} = PD^{-1}P^{-1}}$, where $D^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

a.) $A^{1000} = PD^{1000}P^{-1}$

$$= P \begin{bmatrix} (-1)^{1000} & 0 & 0 \\ 0 & (-1)^{1000} & 0 \\ 0 & 0 & 1^{1000} \end{bmatrix} P^{-1}$$

$$= PIP^{-1} = PP^{-1} = I;$$

d.) $A^{-2301} = (A^{-1})^{2301}$

$= P(D^{-1})^{2301}P^{-1}$

$= P\begin{bmatrix} (-1)^{2301} & 0 & 0 \\ 0 & (-1)^{2301} & 0 \\ 0 & 0 & 1^{2301} \end{bmatrix} P^{-1}$

$= \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$

$= PDP^{-1} = A$