Section 1.4

2.) c.) $(B+C)A = \left(\begin{bmatrix} 0 & 2 \\ 1 & -4 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}\right)\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 4 & 3 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ -18 & -22 \end{bmatrix}$ and

$BA + CA = \begin{bmatrix} 0 & 2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 4 & 8 \\ -5 & -17 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ -13 & -5 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ -18 & -22 \end{bmatrix}$

3.) b.) $(AB)^T = \left(\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & -4 \end{bmatrix}\right)^T$

$= \begin{bmatrix} -1 & 10 \\ 4 & -12 \end{bmatrix}^T = \begin{bmatrix} -1 & 4 \\ 10 & -12 \end{bmatrix}$ and

$B^T A^T = \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 10 & -12 \end{bmatrix}$

5.) $\det A = \det \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix} = 8 - (-12) = 20$, so

$A^{-1} = \frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1/5 & 3/20 \\ -1/5 & 1/10 \end{bmatrix}$

7.) $\det C = \det \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = 6 - 0 = 6$, so

$C^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

9.) $\det\begin{bmatrix} \frac{1}{2}(e^x+e^{-x}) & \frac{1}{2}(e^x-e^{-x}) \\ \frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x+e^{-x}) \end{bmatrix}$

$= \frac{1}{4}(e^x+e^{-x})^2 - \frac{1}{4}(e^x-e^{-x})^2$

$= \frac{1}{4}(e^{2x}+2+e^{-2x}) - \frac{1}{4}(e^{2x}-2+e^{-2x})$

$= \cancel{\frac{1}{4}e^{2x}} + \frac{1}{2} + \cancel{\frac{1}{4}e^{-2x}} - \cancel{\frac{1}{4}e^{2x}} + \frac{1}{2} - \cancel{\frac{1}{4}e^{-2x}}$

$= 1$, so

$$\begin{bmatrix} \frac{1}{2}(e^x+e^{-x}) & \frac{1}{2}(e^x-e^{-x}) \\ \frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x+e^{-x}) \end{bmatrix}^{-1}$$

$$= \frac{1}{1}\begin{bmatrix} \frac{1}{2}(e^x+e^{-x}) & -\frac{1}{2}(e^x-e^{-x}) \\ -\frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x+e^{-x}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(e^x+e^{-x}) & \frac{1}{2}(e^{-x}-e^x) \\ \frac{1}{2}(e^{-x}-e^x) & \frac{1}{2}(e^x+e^{-x}) \end{bmatrix}$$

10.) $\det\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \cos^2\theta + \sin^2\theta = 1$, so

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$$

11.) $$(A^T)^{-1} = \frac{1}{20}\begin{bmatrix} 4 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1/5 & -1/5 \\ 3/20 & 1/10 \end{bmatrix}$$ and

$$(A^{-1})^T = \begin{bmatrix} 1/5 & 3/20 \\ -1/5 & 1/10 \end{bmatrix}^T = \begin{bmatrix} 1/5 & -1/5 \\ 3/20 & 1/10 \end{bmatrix}$$

12.) $$A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 1/5 & 3/20 \\ -1/5 & 1/10 \end{bmatrix},$$ so

$$(A^{-1})^{-1} = \frac{1}{1/50 - (-3/100)}\begin{bmatrix} 1/10 & -3/20 \\ 1/5 & 1/5 \end{bmatrix}$$

$$= 20\begin{bmatrix} 1/10 & -3/20 \\ 1/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$$

15.) $$(7A)^{-1} = \frac{1}{7}A^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix} \rightarrow$$

$$A^{-1} = 7\begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -21 & 49 \\ 7 & -14 \end{bmatrix},$$ then

$$A = (A^{-1})^{-1} = \frac{1}{294 - 343}\begin{bmatrix} -14 & -49 \\ -7 & -21 \end{bmatrix}$$

$$= \frac{-1}{49}\begin{bmatrix} -14 & -49 \\ -7 & -21 \end{bmatrix} = \begin{bmatrix} 14/49 & 1 \\ 7/49 & 21/49 \end{bmatrix}$$

$$= \begin{bmatrix} 2/7 & 1 \\ 1/7 & 3/7 \end{bmatrix}$$

17.) $\begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}^{-1} = \frac{1}{-5-8} \begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} -5/13 & 2/13 \\ 4/13 & 1/13 \end{bmatrix}$,

so $(I+2A) = ((I+2A)^{-1})^{-1}$

$= \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5/13 & 2/13 \\ 4/13 & 1/13 \end{bmatrix} \rightarrow$

$2A = \begin{bmatrix} -5/13 & 2/13 \\ 4/13 & 1/13 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -18/13 & 2/13 \\ 4/13 & -12/13 \end{bmatrix}$

$\rightarrow A = \frac{1}{2} \begin{bmatrix} -18/13 & 2/13 \\ 4/13 & -12/13 \end{bmatrix} = \begin{bmatrix} -9/13 & 1/13 \\ 2/13 & -6/13 \end{bmatrix}$

18.) $A = (A^{-1})^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}^{-1}$

$= \frac{1}{10-(-3)} \begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 5/13 & 1/13 \\ -3/13 & 2/13 \end{bmatrix}$

20.) $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$

a.) $A^3 = A^2 A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} \cdot A$

$= \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$

b.) $A^{-1} = \frac{1}{2-0}\begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -2 & 1 \end{bmatrix}$, so

$A^{-3} = (A^{-1})^3 = (A^{-1})^2 \cdot A^{-1}$

$= \begin{bmatrix} 1/2 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ -2 & 1 \end{bmatrix} \cdot A^{-1} = \begin{bmatrix} 1/4 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1/8 & 0 \\ -7/2 & 1 \end{bmatrix}$ OR

$A^{-3} = (A^3)^{-1} = \frac{1}{8-0}\begin{bmatrix} 1 & 0 \\ -28 & 8 \end{bmatrix} = \begin{bmatrix} 1/8 & 0 \\ -7/2 & 1 \end{bmatrix}$

23.) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then

$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$ and

$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$, so

$\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \rightarrow \boxed{c=0}$ and $\boxed{a=d}$

24.) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, so

$AC = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}$ and

$$CA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}, \text{ then}$$

$$\begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \rightarrow \boxed{b=0} \text{ and } \boxed{a=d}.$$

31.) a.) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then

$$(A+B)(A-B) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

and $A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, i.e.,

$(A+B)(A-B) \neq A^2 - B^2$

b.) $(A+B)(A-B) = A(A-B) + B(A-B)$

$= A^2 - AB + BA - B^2$

c.) Theorem: If $AB = BA$, then

$(A+B)(A-B) = A^2 - B^2$

32.) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$,

$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

all satisfy $A^2 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

33.) a.) $A^2 + 2A + I = 0 \longrightarrow$

$I = -A^2 - 2A = (-A-2)A$ and
$I = -A^2 - 2A = A(-A-2)$, so
A is invertible and $A^{-1} = -A-2$

34.) No, since $A^3 = I \rightarrow (A^2)A = I$
and $A(A^2) = I$, so A is invertible
and $A^{-1} = A^2$

35.) No; if the kth row of A is
all zeroes, then the kth row
of AB (for all B) is all zeroes, so $AB \neq I$.

36.) No; if the mth and kth rows of A
are identical, then the mth and kth
rows of AB (for all B) are identical,
so $AB \neq I$.

37.) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow$

$\begin{cases} a+g=1 \\ a+d=0 \\ d+g=0 \end{cases}$; $\begin{cases} b+h=0 \\ b+e=1 \\ e+h=0 \end{cases}$; $\begin{cases} c+j=0 \\ c+f=0 \\ f+j=1 \end{cases} \longrightarrow$

$$\begin{cases} -g+d=-1 \\ d+g=0 \end{cases} \rightarrow 2d=-1 \rightarrow \boxed{d=-1/2}$$

and $\boxed{g=1/2}$, $\boxed{a=1/2}$;

$$\begin{cases} -h+e=1 \\ e+h=0 \end{cases} \rightarrow 2e=1 \rightarrow \boxed{e=1/2}$$ and

$\boxed{h=-1/2}$, $\boxed{b=1/2}$

$$\begin{cases} -j+f=0 \\ f+j=1 \end{cases} \rightarrow 2f=1 \rightarrow \boxed{f=1/2}$$ and

$\boxed{j=1/2}$, $\boxed{c=-1/2}$; so

$$A^{-1}=\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

40.) $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$

$= (C^{-1})^{-1}A^{-1}\cdot(I)\cdot AD^{-1}$

$= CA^{-1}AD^{-1} = CID^{-1} = CD^{-1}$

41.) $R=[r_1\ r_2 \cdots r_n]$ and $C=\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$:

Then

$$RC = r_1c_1 + r_2c_2 + \cdots + r_nc_n \ ;$$

$$CR = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$$

$$\begin{bmatrix} r_1c_1 & r_2c_1 & & \cdots & r_nc_1 \\ r_1c_2 & r_2c_2 & r_3c_2 & & r_nc_2 \\ r_1c_3 & r_2c_3 & r_3c_3 & r_4c_3 & r_nc_3 \\ \vdots & & & \ddots & \vdots \\ r_1c_n & & \cdots & & r_nc_n \end{bmatrix}$$

$\rightarrow tr(CR) = r_1c_1 + r_2c_2 + \cdots + r_nc_n$

43.) a.) assume A is invertible so A^{-1} exists. If
$AB = AC \rightarrow A^{-1}(AB) = A^{-1}(AC) \rightarrow$
$(A^{-1}A)B = (A^{-1}A)C \rightarrow IB = IC \rightarrow$
$B = C$.

45.) a.) assume A^{-1}, B^{-1}, and $(A+B)^{-1}$ exist. Then
$A(A^{-1}+B^{-1})B(A+B)^{-1}$
$= (AA^{-1} + AB^{-1})B(A+B)^{-1}$
$= (I + AB^{-1})B(A+B)^{-1}$

$$= (B + AB^{-1}B)(A+B)^{-1}$$
$$= (B + AI)(A+B)^{-1}$$
$$= (A+B)(A+B)^{-1} = I$$

b.) (Show $A^{-1}+B^{-1}$ is invertible.)

$$A(A^{-1}+B^{-1})B(A+B)^{-1} = I \rightarrow$$
$$A^{-1}A(A^{-1}+B^{-1})B(A+B)^{-1} = A^{-1}I \rightarrow$$
$$I(A^{-1}+B^{-1})B(A+B)^{-1}A = A^{-1}A \rightarrow$$
$$(A^{-1}+B^{-1}) \cdot \underbrace{B(A+B)^{-1}A} = I$$

and

$$A(A^{-1}+B^{-1})B(A+B)^{-1} = I \rightarrow$$
$$A(A^{-1}+B^{-1})B(A+B)^{-1}(A+B) = I(A+B) \rightarrow$$
$$A(A^{-1}+B^{-1})BI \cdot B^{-1} = (A+B)B^{-1} \rightarrow$$
$$A(A^{-1}+B^{-1})I = (A+B)B^{-1} \rightarrow$$
$$(A+B)^{-1}A(A^{-1}+B^{-1}) = (A+B)^{-1}(A+B)B^{-1} \rightarrow$$
$$B(A+B)^{-1}A(A^{-1}+B^{-1}) = B \cdot IB^{-1} \rightarrow$$
$$\underbrace{B(A+B)^{-1}A} \cdot (A^{-1}+B^{-1}) = I.$$

Thus $A^{-1}+B^{-1}$ is invertible and

$$(A^{-1}+B^{-1})^{-1} = B(A+B)^{-1}A$$

46.) a.) Assume that A is idempotent, i.e., $A^2 = A$. Show $I - A$ is idempotent, i.e., show that $(I-A)^2 = I - A$:

$$(I-A)^2 = (I-A)(I-A)$$
$$= (I-A)I - (I-A)A$$
$$= I^2 - AI - IA + A^2$$
$$= I - A - A + A$$
$$= I - A + (-A + A)$$
$$= I - A + 0 = I - A.$$

b.) Assume that A is idempotent, i.e., $A^2 = A$. Show that $2A - I$ is invertible with

$$(2A - I)^{-1} = 2A - I :$$

$$(2A-I)(2A-I) = (2A-I)2A - (2A-I)I$$
$$= 4A^2 - I2A - 2AI + I^2$$
$$= 4A - 2A - 2A + I$$
$$= 0 + I$$
$$= I.$$

50.) $ABC^TDBA^TC = AB^T \rightarrow$

$A^{-1}ABC^TDBA^TCC^{-1} = A^{-1}AB^TC^{-1} \rightarrow$

$IBC^TDBA^TI = IB^TC^{-1} \rightarrow$

$B^{-1}BC^TDBA^T(A^T)^{-1} = B^{-1}B^TC^{-1}(A^T)^{-1} \rightarrow$

$IC^TDBI = B^{-1}B^TC^{-1}(A^T)^{-1} \rightarrow$

$(C^T)^{-1}C^TDBB^{-1} = (C^T)^{-1}B^{-1}B^TC^{-1}(A^T)^{-1}B^{-1} \rightarrow$

$IDI = (C^T)^{-1}B^{-1}B^TC^{-1}(A^T)^{-1}B^{-1} \rightarrow$

$D = (C^T)^{-1}B^{-1}B^TC^{-1}(A^T)^{-1}B^{-1}$

TRUE/FALSE

(a) F (b) F (c) F (d) F

(e) F (f) T (g) T (h) T

(i) F (j) T (k) F