Section 1.5

1.) a.) T b.) F c.) F d.) F

2.) a.) T b.) T c.) T d.) F

3.) a.) 3 times row 2 added to row 1: $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

b.) $-\frac{1}{7}$ times row 1 : $\begin{bmatrix} -1/7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c.) 5 times row 1 added to row 3 : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$

d.) switch rows 1 and 3: $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5.) a.) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -6 & -6 \\ -1 & -2 & 5 & -1 \end{bmatrix}$

(switch rows 1 and 2)

b.) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ -1 & 9 & 4 & -12 & -10 \end{bmatrix}$

(add -3 times row 2 to row 3.)

e.) $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 13 & 28 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

(Add 4 times row 3 to row 1.)

8.) a.) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b.) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c.) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ d.) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

9.) a.) $\det \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} = 7-8 = -1 \neq 0$ so

A is invertible, then

$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right]$

so $A^{-1} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$

b.) $\det \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} = 16-(16) = 0$, so A is NOT invertible.

10.) a.) $\det \begin{bmatrix} 1 & -5 \\ 3 & -16 \end{bmatrix} = -16-(-15) = -1 \neq 0$

so A is invertible, then

$$\left[\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ 3 & -16 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 16 & -5 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} 16 & -5 \\ 3 & -1 \end{bmatrix}$

b.) $\det \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix} = -12 - (-12) = 0$, so A is NOT invertible.

11.) a.) $$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right], \text{ so } A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

12.) b.) $$\left[\begin{array}{ccc|ccc} 1/5 & 1/5 & -2/5 & 1 & 0 & 0 \\ 2/5 & -3/5 & -3/10 & 0 & 1 & 0 \\ 1/5 & -4/5 & 1/10 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 2 & -3 & -3/2 & 0 & 5 & 0 \\ 1 & -4 & 1/2 & 0 & 0 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & -5 & 5/2 & -10 & 5 & 0 \\ 0 & -5 & 5/2 & -5 & 0 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & -5 & 5/2 & -10 & 5 & 0 \\ 0 & 0 & 0 & 5 & -5 & 5 \end{array}\right]$$

(row of zeroes) so A is not invertible.

13.) $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right], \text{ so } A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

15.) $$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 7/2 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right], \text{ so } A^{-1} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

18.) $$\left[\begin{array}{cccc|cccc} 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 5 & -3 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 5 & -5 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 8 & -5 & 0 & -2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -5 & -4 & -2 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3/2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 4/5 & 2/5 & -1/5 & -1/5 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -4/5 & 3/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 3/2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 4/5 & 2/5 & -1/5 & -1/5 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -4/5 & 3/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 3/2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4/5 & 2/5 & -1/5 & -1/5 \end{array}\right], \text{ so } A^{-1} = \begin{bmatrix} -4/5 & 3/5 & 1/5 & 1/5 \\ 3/2 & 0 & -1 & 0 \\ 1/2 & 0 & 0 & 0 \\ 4/5 & 2/5 & -1/5 & -1/5 \end{bmatrix}$$

19.) a.) $$A^{-1} = \begin{bmatrix} 1/k_1 & 0 & 0 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 0 & 0 & 1/k_4 \end{bmatrix}$$

b.) $$\left[\begin{array}{cccc|cccc} k & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} k & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/k & -1/k & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/k & -1/k \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right], \text{ so } A^{-1} = \begin{bmatrix} 1/k & -1/k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/k & -1/k \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

20.) b.)

$$\left[\begin{array}{cccc|cccc} k & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & k & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & k & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & k & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 0 & -k^2 & 0 & 0 & 1 & -k & 0 & 0 \\ 1 & 0 & -k^2 & 0 & 0 & 1 & -k & 0 \\ 0 & 1 & 0 & -k^2 & 0 & 0 & 1 & -k \\ 0 & 0 & 1 & k & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & -k^4 & 1 & -k & k^2 & -k^3 \\ 1 & 0 & 0 & k^3 & 0 & 1 & -k & -k^2 \\ 0 & 1 & 0 & -k^2 & 0 & 0 & 1 & -k \\ 0 & 0 & 1 & k & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & -1/k^4 & 1/k^3 & -1/k^2 & 1/k \\ 1 & 0 & 0 & k^3 & 0 & 1 & -k & k^2 \\ 0 & 1 & 0 & -k^2 & 0 & 0 & 1 & -k \\ 0 & 0 & 1 & k & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & -1/k^4 & 1/k^3 & -1/k^2 & 1/k \\ 1 & 0 & 0 & 0 & 1/k & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/k^2 & 1/k & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/k^3 & -1/k^2 & 1/k & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/k & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/k^2 & 1/k & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/k^3 & -1/k^2 & 1/k & 0 \\ 0 & 0 & 0 & 1 & -1/k^4 & 1/k^3 & -1/k^2 & 1/k \end{array}\right], \text{ so}$$

$$A^{-1} = \begin{bmatrix} 1/k & 0 & 0 & 0 \\ -1/k^2 & 1/k & 0 & 0 \\ 1/k^3 & -1/k^2 & 1/k & 0 \\ -1/k^4 & 1/k^3 & -1/k^2 & 1/k \end{bmatrix}$$

21.) $\begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & c-c^2 \\ 0 & c-1 & 0 \\ 1 & 1 & c \end{bmatrix}$ (so $c-1 \neq 0 \rightarrow \boxed{c \neq 1}$ and

$c - c^2 = c(1-c) \neq 0$ so $\boxed{c \neq 0}$)

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & c \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

22.) $\begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix} \sim \begin{bmatrix} 0 & 1-c^2 & -c \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & c(c^2-1)-c \\ 1 & 0 & 1-c^2 \\ 0 & 1 & c \end{bmatrix}$

($c(c^2-1)-c = c(c^2-2) \neq 0$, so $\boxed{c \neq 0}$

and $\boxed{c \neq \pm\sqrt{2}}$)

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

29.) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$. Show at least one entry in the

be an elementary matrix.

third row must be zero. assume that ALL entries in row 3 are NONZERO (This is a proof by contradiction.). Since $a \neq 0$ then A is created by multiplying the first row of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by a and adding it to the 3rd row. Since $b \neq 0$, then A is created by multiplying the 2nd row of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by b and adding it to the 3rd row. This contradicts the fact that A is elementary since TWO operations were applied to I to create A. Thus, at least one of a, b, c is zero!

30.) Consider matrix
$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}.$$
Show that A is not invertible.

Assume that A IS invertible.
(This is a proof by contradiction.)
Thus $a \neq 0$ and $h \neq 0$ (to avoid rows of zeroes) and

$$\begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$; but the row of zeroes is a contradiction

since we assumed A is invertible.
Thus A is not invertible.

32.) Assume A is invertible and B is row equivalent to A. Show B is invertible: Since B is row equivalent to A, then

$$B = E_k \cdots E_2 E_1 A,$$ where

$E_1, E_2, \ldots,$ and E_k are elementary matrices, all of which are invertible. Then B is invertible with

$$B^{-1} = A^{-1}E_1^{-1}E_2^{-1}\cdots E_k^{-1},$$

since

$$B(A^{-1}E_1^{-1}E_2^{-1}\cdots E_k^{-1})$$

$$= (E_k \cdots E_2 E_1 A)(A^{-1}E_1^{-1}E_2^{-1}\cdots E_k^{-1}) \qquad (AA^{-1}=I)$$

$$= E_k \cdots E_2 E_1 \cdot E_1^{-1}E_2^{-1}\cdots E_k^{-1} \qquad (E_1E_1^{-1}=I)$$

$$= E_k \cdots E_2 \cdot E_2^{-1} \cdots E_k^{-1} \qquad (E_2E_2^{-1}=I)$$

$$= \cdots = E_kE_k^{-1} = I.$$

TRUE/FALSE

(a) F (b) T (c) T (d) T

(e) T (f) T (g) F