Section 1.6

1.) $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -5 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & -1 \\ 0 & 1 & -5 & 1 \end{array}\right]$,

then $\begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$ has solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

i.e., $x_1 = 3$, $x_2 = -1$

4.) $\left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} -1 & -3 & -2 & 1 & -2 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & -1 & 2 & 0 \\ 0 & -6 & -4 & 3 & -5 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 2 & -3 \\ 0 & 0 & 2 & 3 & -5 & 6 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 3/2 & -5/2 & 3 \\ 0 & 1 & 0 & -3/2 & 5/2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & -3/2 & 5/2 & -2 \\ 0 & 0 & 1 & 3/2 & -5/2 & 3 \end{array}\right]$,

so solution to system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}$$

5.) $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -4 & 0 & 1 & 0 \\ -4 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & -1 & 1 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \\ 0 & 1 & 1 & 4/5 & 0 & 1/5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \\ 0 & 1 & 0 & 3/5 & 1/5 & 1/5 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & -1/5 \\ 0 & 1 & 0 & 3/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right]$, so solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/5 & 0 & -1/5 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & -1/5 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}, \text{ i.e.,}$$

$x_1 = 1,\ x_2 = 5,\ x_3 = -1$

6.) $\left[\begin{array}{cccc|cccc} 0 & -1 & -2 & -3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 1 & 3 & 7 & 9 & 0 & 0 & 1 & 0 \\ -1 & -2 & -4 & -6 & 0 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cccc|cccc} 0 & -1 & -2 & -3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 5 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{cccc|cccc} 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 1 & 3 & -2 & 2 & 0 \\ 1 & 0 & 0 & -1 & 5 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 6 & -3 & 4 & 1 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 6 & -3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array}\right],$$ so solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 6 & -3 & 4 & 1 \\ 1 & 0 & 1 & 1 \\ -3 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 7 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 10 \\ -7 \end{bmatrix},$$ i.e.,

$$x_1 = -6,\ x_2 = 1,\ x_3 = 10,\ x_4 = -7$$

7.) $\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 0 & -1 & 1 & -3 \\ 1 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 0 & 1 & -1 & 3 \\ 1 & 0 & 2 & -5 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right]$, so solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 - 5b_2 \\ -b_1 + 3b_2 \end{bmatrix}, \text{ i.e.,}$$

$$x_1 = 2b_1 - 5b_2, \quad x_2 = 3b_2 - b_1$$

9.) $\left[\begin{array}{cc|c} 1 & -5 & b_1 \\ 3 & 2 & b_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -5 & b_1 \\ 0 & 17 & -3b_1 + b_2 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & -5 & b_1 \\ 0 & 1 & -\frac{3}{17} b_1 + \frac{1}{17} b_2 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 0 & \frac{2}{17} b_1 + \frac{5}{17} b_2 \\ 0 & 1 & -\frac{3}{17} b_1 + \frac{1}{17} b_2 \end{array}\right] \rightarrow$

$$x_1 = \frac{2}{17} b_1 + \frac{5}{17} b_2, \quad x_2 = \frac{-3}{17} b_1 + \frac{1}{17} b_2$$

i.) $b_1 = 1, \ b_2 = 4 \longrightarrow x_1 = \frac{22}{17}, \ x_2 = \frac{1}{17}$

ii.) $b_1 = -2, \ b_2 = 5 \longrightarrow x_1 = \frac{21}{17}, \ x_2 = \frac{11}{17}$

12.) $\left[\begin{array}{ccc|c} 1 & 3 & 5 & b_1 \\ -1 & -2 & 0 & b_2 \\ 2 & 5 & 4 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & b_1 \\ 0 & 1 & 5 & b_1+b_2 \\ 0 & 1 & 4 & 2b_2+b_3 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -10 & -3(b_1+b_2)+b_1 \\ 0 & 1 & 5 & b_1+b_2 \\ 0 & 0 & -1 & -(b_1+b_2)+2b_2+b_3 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -10 & -2b_1-3b_2 \\ 0 & 1 & 5 & b_1+b_2 \\ 0 & 0 & -1 & -b_1+b_2+b_3 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10(-b_1+b_2+b_3)+-2b_1-3b_2 \\ 0 & 1 & 0 & 5(-b_1+b_2+b_3)+b_1+b_2 \\ 0 & 0 & 1 & b_1-b_2-b_3 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8b_1-13b_2-10b_3 \\ 0 & 1 & 0 & -4b_1+6b_2+5b_3 \\ 0 & 0 & 1 & b_1-b_2-b_3 \end{array}\right] \rightarrow$

$x_1 = 8b_1 - 13b_2 - 10b_3$,
$x_2 = -4b_1 + 6b_2 + 5b_3$,
$x_3 = b_1 - b_2 - b_3$

i.) $b_1=1, b_2=0, b_3=-1 \rightarrow x_1=18, x_2=-9, x_3=2$

ii.) $b_1=0, b_2=1, b_3=1 \rightarrow x_1=-23, x_2=11, x_3=-2$

iii.) $b_1=-1, b_2=-1, b_3=0 \rightarrow x_1=5, x_2=-2, x_3=0$

13.) $\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ -2 & 1 & b_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 7 & 2b_1 + b_2 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 1 & \frac{2}{7}b_1 + \frac{1}{7}b_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -3\left(\frac{2}{7}b_1 + \frac{1}{7}b_2\right) + b_1 \\ 0 & 1 & \frac{2}{7}b_1 + \frac{1}{7}b_2 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{7}b_1 - \frac{3}{7}b_2 \\ 0 & 1 & \frac{2}{7}b_1 + \frac{1}{7}b_2 \end{array}\right] \rightarrow x_1 = \frac{1}{7}b_1 - \frac{3}{7}b_2,$ $x_2 = \frac{2}{7}b_1 + \frac{1}{7}b_2$

for all choices of b_1, b_2

14.) $\left[\begin{array}{cc|c} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & -2b_2 + b_1 \\ 3 & -2 & b_2 \end{array}\right] \rightarrow$ for solution

$-2b_2 + b_1 = 0 \rightarrow \boxed{b_1 = 2b_2} \rightarrow$

$\sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & \frac{2}{3} & \frac{1}{3}b_2 \end{array}\right] \rightarrow x_1 + \frac{2}{3}x_2 = \frac{1}{3}b_2 \rightarrow$

$x_2 = t$ any #, $x_1 = \frac{1}{3}b_2 - \frac{2}{3}t$

15.) $\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & -3 & 12 & 3b_1 + b_3 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & 0 & 0 & (-4b_1 + b_2) + (3b_1 + b_3) \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 1 & -4 & -\frac{4}{3}b_1 + \frac{1}{3}b_2 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right] \rightarrow \text{for solution}$$

$-b_1 + b_2 + b_3 = 0 \rightarrow \boxed{b_1 = b_2 + b_3} \rightarrow$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2\left(-\frac{4}{3}b_1 + \frac{1}{3}b_2\right) + b_1 \\ 0 & 1 & -4 & -\frac{4}{3}b_1 + \frac{1}{3}b_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -\frac{5}{3}b_1 + \frac{2}{3}b_2 \\ 0 & 1 & -4 & -\frac{4}{3}b_1 + \frac{1}{3}b_2 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$$

$$\begin{cases} x_1 - 3x_3 = \frac{-5}{3}b_1 + \frac{2}{3}b_2 \\ x_2 - 4x_3 = \frac{-4}{3}b_1 + \frac{1}{3}b_2 \end{cases} \rightarrow \text{let } x_3 = t$$

any # $\rightarrow$ $x_1 = 3t - \frac{5}{3}b_1 + \frac{2}{3}b_2$,

$x_2 = 4t - \frac{4}{3}b_1 + \frac{1}{3}b_2$

17) $\left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1 + b_2 \\ 0 & -1 & 11 & 5 & 3b_1 + b_3 \\ 0 & 1 & -11 & -5 & -4b_1 + b_4 \end{array}\right]$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -8 & -3 & (-4b_1 + b_4) + b_1 \\ 0 & 0 & 0 & 0 & (-4b_1 + b_4) + 2b_1 + b_2 \\ 0 & 0 & 0 & 0 & (-4b_1 + b_4) + 3b_1 + b_3 \\ 0 & 1 & -11 & -5 & -4b_1 + b_4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -8 & -3 & -3b_1 + b_4 \\ 0 & 0 & 0 & 0 & -2b_1 + b_2 + b_4 \\ 0 & 0 & 0 & 0 & -b_1 + b_3 + b_4 \\ 0 & 1 & -11 & -5 & -4b_1 + b_4 \end{array}\right] \rightarrow \text{for solution}$$

$-2b_1 + b_2 + b_4 = 0 \rightarrow \boxed{b_4 = 2b_1 - b_2}$;

$-b_1 + b_3 + b_4 = 0 \rightarrow \boxed{b_4 = b_1 - b_3}$

$\rightarrow 2b_1 - b_2 = b_1 - b_3 \rightarrow$

$\boxed{b_1 - b_2 + b_3 = 0}$ and $\boxed{b_4 = r \text{ any } \#}$;

$$\begin{cases} x_1 - 8x_3 - 3x_4 = -3b_1 + b_4 \\ x_2 - 11x_3 - 5x_4 = -4b_1 + b_4 \end{cases}, \text{ so let}$$

$x_4 = t$ any #, $x_3 = s$ any # $\rightarrow$

$x_1 = 8s + 3t - 3b_1 + b_4$

$x_2 = 11s + 5t - 4b_1 + b_4$

18.) b.) $Ax = 4x = 4Ix \rightarrow Ax - 4Ix = 0 \rightarrow$
$(A - 4I)x = 0 \rightarrow$

$$\begin{bmatrix} -2 & 1 & 2 \\ 2 & -2 & -2 \\ 3 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 2 & -2 & -2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rightarrow \boxed{x_2 = 0}$ and $x_1 - x_3 = 0 \rightarrow \boxed{x_3 = t}$

any # and $\boxed{x_1 = t}$

20.) $$\left[\begin{array}{ccc|ccccc} 1 & -1 & 1 & 2 & -1 & 5 & 7 & 8 \\ 2 & 3 & 0 & 4 & 0 & -3 & 0 & 1 \\ 0 & 2 & -1 & 3 & 5 & -7 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccccc} 1 & -1 & 1 & 2 & -1 & 5 & 7 & 8 \\ 0 & 5 & -2 & 0 & 2 & -13 & -14 & -15 \\ 0 & 2 & -1 & 3 & 5 & -7 & 2 & 1 \end{array}\right] \leftarrow (x)(-2)(+)$$

$$\sim \left[\begin{array}{ccc|ccccc} 1 & -1 & 1 & 2 & -1 & 5 & 7 & 8 \\ 0 & 1 & 0 & -6 & -8 & 1 & -18 & -17 \\ 0 & 2 & -1 & 3 & 5 & -7 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccccc} 1 & 0 & 1 & -4 & -9 & 6 & -11 & -9 \\ 0 & 1 & 0 & -6 & -8 & 1 & -18 & -17 \\ 0 & 0 & -1 & 15 & 21 & -9 & 38 & 35 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 11 & 12 & -3 & 27 & 26 \\ 0 & 1 & 0 & -6 & -8 & 1 & -18 & -17 \\ 0 & 0 & 1 & -15 & -21 & 9 & -38 & -35 \end{array}\right]$$, where each column

represents $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

22.) assume $Ax=0$ has solution $x=0$ and assume Q is invertible. Show $(QA)x=0$ has solution $x=0$:

$(QA)x=0 \rightarrow Q^{-1}(QA)x = Q^{-1}0 \rightarrow$

$(Q^{-1}Q)Ax=0 \rightarrow IAx=0 \rightarrow Ax=0$

$\rightarrow x=0$

23.) assume x_1 is a solution to $Ax=b$ and assume x_0 is a solution to $Ax=0$. Let x_2 be any solution to $Ax=b$. Show that $x_2 = x_1 + x_0$:

Since x_1 and x_2 are solutions to $Ax=b$, i.e., $Ax_1=b$ and $Ax_2=b \rightarrow$

$Ax_2 - Ax_1 = b-b=0 \rightarrow A(x_2-x_1)=0$

$\rightarrow x_2-x_1$ solves $Ax=0 \rightarrow x_2-x_1=x_0$

$\rightarrow x_2 = x_1 + x_0$.

TRUE/FALSE

(a) T (b) T (c) T (d) T

(e) T (f) T (g) T