Section 1.7

7.) $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \longrightarrow$

$A^2 = \begin{bmatrix} 1^2 & 0 \\ 0 & (-2)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$;

$A^{-2} = (A^2)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$;
$= (A^{-1})^2$

$A^{-k} = (A^{-1})^k = \begin{bmatrix} 1^k & 0 \\ 0 & (-1/2)^k \end{bmatrix}$

10.) $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -1/2 & 0 & 0 & 0 \\ 0 & -1/4 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \rightarrow$

$A^2 = \begin{bmatrix} (-2)^2 & 0 & 0 & 0 \\ 0 & (-4)^2 & 0 & 0 \\ 0 & 0 & (-3)^2 & 0 \\ 0 & 0 & 0 & (2)^2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$;

$A^{-2} = (A^2)^{-1} = \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/16 & 0 & 0 \\ 0 & 0 & 1/9 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$;
$= (A^{-1})^2$

$$A^{-k} = (A^{-1})^k = \begin{bmatrix} (-\tfrac{1}{2})^k & 0 & 0 & 0 \\ 0 & (-\tfrac{1}{4})^k & 0 & 0 \\ 0 & 0 & (-\tfrac{1}{3})^k & 0 \\ 0 & 0 & 0 & (\tfrac{1}{2})^k \end{bmatrix}$$

12.) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} (-1)(3)(5) & 0 & 0 \\ 0 & (2)(5)(-2) & 0 \\ 0 & 0 & (4)(7)(3) \end{bmatrix} = \begin{bmatrix} -15 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 84 \end{bmatrix}$$

13.) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{39} = \begin{bmatrix} 1^{39} & 0 \\ 0 & (-1)^{39} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

14.) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{1000} = \begin{bmatrix} 1^{1000} & 0 \\ 0 & (-1)^{1000} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

19.) NO 20.) YES 21.) YES

25.) $A = \begin{bmatrix} 4 & -3 \\ a+5 & -1 \end{bmatrix} \rightarrow a+5 = -3 \rightarrow a = -8$

26.) $A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$

27.) A is invertible for $x \neq 1$, $x \neq -2$, $x \neq 4$

30.) assume A is symmetric and B is a matrix

a.) Show $B^T B$ is symmetric:

$$(B^T B)^T = B^T (B^T)^T = B^T B$$

b.) Show $B B^T$ is symmetric:

$$(B B^T)^T = (B^T)^T B^T = B B^T$$

c.) Show $B^T A B$ is symmetric:

$$(B^T A B)^T = B^T A^T (B^T)^T \quad (A \text{ is symmetric})$$

$$= B^T A B$$

31.) Choose $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

32.) Choose $A = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$, so

$$A^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

34.) Assume A is symmetric, i.e., $A^T = A$.

a.) Show A^2 is symmetric:

$$(A^2)^T = (A^T)^2 = (A)^2 = A^2$$

b.) Show $2A^2 - 3A + I$ is symmetric:

$$(2A^2 - 3A + I)^T = (2A^2)^T - (3A)^T + I^T$$
$$= 2(A^2)^T - 3A^T + I$$
$$= 2(A^T)^2 - 3A + I$$
$$= 2A^2 - 3A + I$$

36.) $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then

$A^2 - 3A - 4I = 0 \longrightarrow$

$$\begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix} - 3\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} a^2 - 3a - 4 & 0 & 0 \\ 0 & b^2 - 3b - 4 & 0 \\ 0 & 0 & c^2 - 3c - 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow a^2-3a-4=(a-4)(a+1)=0 \rightarrow$

$\boxed{a=4}$, $\boxed{a=-1}$;

$\rightarrow b^2-3b-4=(b-4)(b+1)=0 \rightarrow$

$\boxed{b=4}$, $\boxed{b=-1}$;

$\rightarrow c^2-3b-4=(c-4)(c+1)=0 \rightarrow$

$\boxed{c=4}$, $\boxed{c=-1}$ so solutions are:

$$\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}, \begin{bmatrix}4&0&0\\0&4&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}4&0&0\\0&-1&0\\0&0&4\end{bmatrix}, \begin{bmatrix}4&0&0\\0&-1&0\\0&0&-1\end{bmatrix},$$

$$\begin{bmatrix}-1&0&0\\0&4&0\\0&0&4\end{bmatrix}, \begin{bmatrix}-1&0&0\\0&4&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}-1&0&0\\0&-1&0\\0&0&4\end{bmatrix}, \begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$

39.) Solve $\begin{bmatrix}a&b\\0&c\end{bmatrix}^3=\begin{bmatrix}1&30\\0&-8\end{bmatrix}$:

$$\begin{bmatrix}a&b\\0&c\end{bmatrix}\begin{bmatrix}a&b\\0&c\end{bmatrix}\begin{bmatrix}a&b\\0&c\end{bmatrix}=\begin{bmatrix}a^2&ab+bc\\0&c^2\end{bmatrix}\begin{bmatrix}a&b\\0&c\end{bmatrix}$$

$$=\begin{bmatrix}a^3&a^2b+abc+bc^2\\0&c^3\end{bmatrix}=\begin{bmatrix}1&30\\0&-8\end{bmatrix}\rightarrow$$

$a^3=1 \rightarrow \boxed{a=1}$ and $c^3=-8 \rightarrow \boxed{c=-2}$

and $a^2b+abc+bc^2=b-2b+4b$

$= 3b = 30 \rightarrow \boxed{b=10}$

47.) Assume that $A^TA = A$.

a.) Show that A is symmetric, i.e., show $A^T = A$:

$$A^T = (A^TA)^T = A^T(A^T)^T$$
$$= A^TA = A$$

b.) Show that $A^2 = A$:

$$A^2 = AA = A^TA = A$$

TRUE/FALSE

(a) T (b) F (c) F (d) T
(e) T (f) F (g) F (h) T
(i) T (j) F (k) F (l) F
(m) T