Section 9.1

1.) $\begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$\underset{A}{\begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

a.) Solve $L\vec{y} = \vec{b}$ for $\vec{y}$:

$\left[\begin{array}{cc|c} 3 & 0 & 0 \\ -2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$, so $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

b.) Solve $U\vec{x} = \vec{y}$ for $\vec{x}$:

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$, so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\rightarrow x_1 = 2,\ x_2 = 1$

2.) $$\underset{A}{\begin{bmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} -3 \\ -22 \\ 3 \end{bmatrix}}$$:

a.) Solve $L\vec{y} = \vec{b}$ for $\vec{y}$:

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & -3 \\ 2 & 4 & 0 & -22 \\ -4 & -1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 4 & 0 & -20 \\ 0 & -1 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 2 & -6 \end{array}\right]$$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -3 \end{array}\right]$, so $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -3 \end{bmatrix}$.

b.) Solve $U\vec{x} = \vec{y}$ for $\vec{x}$:

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right], \text{ so}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \rightarrow x_1 = -2,\ x_2 = 1,\ x_3 = -3$$

3.) $A = \begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix} \longrightarrow E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \longrightarrow E_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \longrightarrow E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$

$\sim \underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{U}$ then

$$E_3 E_2 E_1 A = U \rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} U$$

$$= \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} U = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} U$$

$$= \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}}_{A} ;$$

Solve $\underbrace{\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -2 \\ -2 \end{bmatrix}}_{\vec{b}}$:

a.) Solve $L\vec{y} = \vec{b}$ for $\vec{y}$:

$$\left[\begin{array}{cc|c} 2 & 0 & -2 \\ -1 & 3 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 3 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \end{array}\right], \text{ so}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

b.) Solve $U\vec{x} = \vec{y}$ for $\vec{x}$:

$$\left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right], \text{ so } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

$\rightarrow x_1 = 3,\ x_2 = -1$

4.) $A = \begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix} \rightarrow E_1 = \begin{bmatrix} -1/5 & 0 \\ 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix} \rightarrow E_2 = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix}$

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1/7 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{U}$$

then

$$E_3 E_2 E_1 A = U \rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$= \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} U$$

$$= \begin{bmatrix} -5 & 0 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} U = \underbrace{\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}}_{A}$$

Solve $\underbrace{\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -10 \\ 19 \end{bmatrix}}_{\vec{b}}$:

a.) Solve $L\vec{y} = \vec{b}$ for $\vec{y}$:

$$\left[\begin{array}{cc|c} -5 & 0 & -10 \\ 6 & -7 & 19 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & -7 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right], \text{ so}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

b.) Solve $U\vec{x} = \vec{y}$ for $\vec{x}$:

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right], \text{ so } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\rightarrow x_1 = 4, \quad x_2 = -1$$

5.) $A = \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}$

$$E_1 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

U ~~then~~

$$E_5 E_4 E_3 E_2 E_1 A = U \rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} U$$

$$= \begin{bmatrix} 2&0&0\\0&1&0\\0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&1&0\\-1&0&1 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0\\0&1&0\\0&-2&1 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&-2&0\\0&0&1 \end{bmatrix} \cdot E_5^{-1} U$$

$$= \begin{bmatrix} 2&0&0\\0&1&0\\-1&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&-2&0\\0&4&1 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0\\0&1&0\\0&0&5 \end{bmatrix} U$$

$$= \begin{bmatrix} 2&0&0\\0&-2&0\\-1&4&1 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&1&0\\0&0&5 \end{bmatrix} U$$

$$= \underbrace{\begin{bmatrix} 2&0&0\\0&-2&0\\-1&4&5 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1&-1&-1\\0&1&-1\\0&0&1 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 2&-2&-2\\0&-2&2\\-1&5&2 \end{bmatrix}}_{A}.$$

Solve $\underbrace{\begin{bmatrix} 2&-2&-2\\0&-2&2\\-1&5&2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1\\x_2\\x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2&0&0\\0&-2&0\\-1&4&5 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1&-1&-1\\0&1&-1\\0&0&1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1\\x_2\\x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -4\\-2\\6 \end{bmatrix}}_{\vec{b}}$:

a.) Solve $L\vec{y} = \vec{b}$ for $\vec{y}$:

$$\left[\begin{array}{ccc|c} 2&0&0&-4\\0&-2&0&-2\\-1&4&5&6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1&0&0&-2\\0&1&0&1\\0&4&5&4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1&0&0&-2\\0&1&0&1\\0&0&5&0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right], \text{ so } \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

b.) Solve $U\vec{x} = \vec{y}$ for $\vec{x}$:

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \rightarrow x_1 = -1,\ x_2 = 1,\ x_3 = 0$

6.)

$$A = \begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} -1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 2 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \underset{U}{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

then

$$E_4 E_3 E_2 E_1 A = U \rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} U$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} U$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} U$$

$$= \underset{L}{\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{A}{\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}}.$$

Solve $$\underset{A}{\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{L}{\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}}:$$

a.) Solve $L\vec{y}=\vec{b}$ for $\vec{y}$:

$$\left[\begin{array}{ccc|c} -3 & 0 & 0 & -33 \\ 1 & 2 & 0 & 7 \\ 0 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 2 & 0 & -4 \\ 0 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right],$$

so $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ 1 \end{bmatrix}$.

b.) Solve $U\vec{x}=\vec{y}$ for $\vec{x}$:

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 11 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right],$$

so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rightarrow x_1=1,\ x_2=-2,\ x_3=1$

7.) a.) $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$ then

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -3 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & 1 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right],$$

so $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$; $U = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix}$ then

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 0 & 4 & -5 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & -1/2 \\ 0 & 4 & 0 & 0 & 1 & 5/6 \\ 0 & 0 & 1 & 0 & 0 & 1/6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1/4 & -7/24 \\ 0 & 1 & 0 & 0 & 1/4 & 5/24 \\ 0 & 0 & 1 & 0 & 0 & 1/6 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/8 & -7/48 \\ 0 & 1 & 0 & 0 & 1/4 & 5/24 \\ 0 & 0 & 1 & 0 & 0 & 1/6 \end{array}\right],$$

so $U^{-1} = \begin{bmatrix} 1/2 & 1/8 & -7/48 \\ 0 & 1/4 & 5/24 \\ 0 & 0 & 1/6 \end{bmatrix}$

b.) $A = LU \rightarrow A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$

$$= \begin{bmatrix} 1/2 & 1/8 & -7/48 \\ 0 & 1/4 & 5/24 \\ 0 & 0 & 1/6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5/48 & -1/48 & -7/48 \\ -7/24 & 11/24 & 5/24 \\ 1/6 & 1/6 & 1/6 \end{bmatrix},$$

so $A^{-1} = \begin{bmatrix} 5/48 & -1/48 & -7/48 \\ -7/24 & 11/24 & 5/24 \\ 1/6 & 1/6 & 1/6 \end{bmatrix}$

9.) a.)

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

then

$$E_3 E_2 E_1 A = U \rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U$$

$$= \underset{L}{\begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}}$$

b.) assume $\begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix}$

$= \begin{bmatrix} d & 0 & 0 \\ ad & e & 0 \\ bd & ce & f \end{bmatrix} \rightarrow \boxed{d=2}$, $\boxed{e=1}$, $\boxed{f=1}$ and

$ad = a(2) = -2 \rightarrow \boxed{a=-1}$,

$bd = b(2) = 2 \rightarrow \boxed{b=1}$, $ce = c(1) = 0$

$\rightarrow \boxed{c=0}$, ~~so~~

$$\underset{A}{\begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}} = \underset{L_1}{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{U_1}{\begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

c.) $$\underset{A}{\begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}} = \underset{L_2}{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{U_2}{\begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

10.) a.) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (make it upper triangular)

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longleftarrow E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\quad U$ then

$E_1 A = U \rightarrow A = E_1^{-1} U$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} U$

BUT NOT LOWER TRIANGULAR, so A has no LU-decomposition

13.) $A = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix} \longrightarrow E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \longrightarrow E_2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \longrightarrow E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}$

$\sim \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{U}$ then

$E_3 E_2 E_1 A = U \rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} U$

$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} U$

$= \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} U$

$= \underbrace{\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{U} = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}$;

Assume $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} b & 0 \\ ab & c \end{bmatrix}$

$\rightarrow \boxed{b=2}$, $\boxed{c={}^{-}3}$, and

$ab = a(2) = 4 \rightarrow \boxed{a=2}$, then

$$\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

18.) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and assume $a \neq 0$.

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e & 1 \end{bmatrix} \begin{bmatrix} r & s \\ 0 & t \end{bmatrix} = \begin{bmatrix} r & s \\ er & es+t \end{bmatrix}$

$\rightarrow \boxed{r=a}$ and $er = c \rightarrow ea = c$

$\rightarrow \boxed{e = c/a}$, so $L = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix}$; and

$\boxed{s=b}$ and $es+t = d \rightarrow \left(\frac{c}{a}\right)(b) + t = d$

$\rightarrow t = d - \frac{bc}{a} \rightarrow \boxed{t = \frac{ad-bc}{a}}$, so

the unique LU-decomposition is

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{bmatrix}}.$$

TRUE/FALSE

(a) F (b) F (c) T